The Pseudospin and Spin Symmetric Dirac Equation for

$PT$-Symmetric Morse and Pöschl-Teller Potentials

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Abstract

We investigate the energy spectra and corresponding Dirac spinors of the Dirac equation for $PT$-symmetric Morse and Pöschl-Teller potentials under the exact pseudo-spin, and spin symmetry. We use the parametric generalization of the Nikiforov-Uvarov method to obtain the energy eigenvalue equations analytically, and corresponding wave functions for the value of the spin-orbit quantum number $\kappa = 0$.

Keywords: Pseudospin symmetry, spin symmetry, $PT$-symmetry, Morse potential, Pöschl-Teller potential, Dirac equation, Nikiforov-Uvarov Method

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I. INTRODUCTION

The pseudospin concept has been studied firstly in spherical nuclei [1, 2]. It is experimentally observed that the single-particle levels labelled as pseudospin doublets are very close in energy [3]. The pseudospin symmetry is discussed in deformed, and exotic nuclei as well [3, 4], and also relevant symmetry in the case of triaxiality [5]. The pseudospin doublets in nuclei are labelled as $(n_r, \ell, j = \ell + 1/2)$, and $(n_r - 1, \ell + 1, j = \ell + 3/2)$, where $n_r$ radial, $\ell$ orbital and $j$ are total angular quantum numbers. The total angular momentum is defined in terms of the pseudo orbital angular momentum, $\tilde{\ell} = \ell + 1$, and the pseudospin, $\tilde{s}$, quantum numbers as $j = \tilde{\ell} + \tilde{s}$ [6]. The pseudospin doublets occur in nuclei, when the sum of the scalar, $V_s(r)$, and vector, $V_v(r)$, potentials are nearly equal to zero [7]. The pseudospin symmetry is exact symmetry in real nuclei, when the derivative of the sum between scalar, and vector potentials is equal to zero. This condition gives a good symmetry for exotic nuclei [3, 8]. The pseudospin concept has been used some phenomena in nuclear theory, such as deformation, and super deformation [9, 10, 11], and to construct an effective shell-model coupling scheme [12].

The solutions of the Klein-Gordon and Dirac equations including spin-orbit coupling term have received a great attentions for different potentials, such as Morse potential [13-16], Pöschl-Teller potential [17-19], Woods-Saxon potential [20], Eckart potential [21-24], harmonic oscillator [25, 26], three parameter diatomic potential [27], and angular dependent potential [28].

The investigation of relativistic and/or non-relativistic quantum systems has been a great interest within the $PT$-symmetric quantum mechanics. Following the earlier work of Bender, and Boettcher [29], the quantum systems under the effects of non-Hermitian $PT$-symmetric and pseudo-Hermitian potentials have been studied in details by many authors by using different methods [29-67]. The $PT$-symmetric formalism has been widely used in quantum fields theories [68-70] and nuclear theory [71], etc. Therefore, our aim in this work is to investigate the bound-state solutions, and the corresponding unnormalized Dirac spinors for $PT$-symmetric generalized Morse and $PT$-symmetric $q$-deformed Pöschl-Teller potentials with exact pseudo-spin, and spin symmetry. We solve the Dirac equation by using a new scheme of the Nikiforov-Uvarov method [72], and give the energy eigenvalue equations and the corresponding wave functions for the value of spin-orbit quantum number $\kappa = 0$. 

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The organization of the present work is as follows. In Section II, we give the Dirac equation with spin-orbit coupling term. In Section III, we construct briefly the new scheme of the NU-method called parametric generalization of the method. In Section IV, we compute the energy eigenvalue equation, and corresponding Dirac spinors of the above potentials in the case of exact pseudo-spin, and spin symmetry. We give our conclusions in Section V.

II. THE DIRAC EQUATION WITH SPIN-ORBIT COUPLING

The Dirac equation for a fermion with mass \( m \) moving in an external scalar and vector potentials reads (\( \hbar = c = 1 \))

\[
[\alpha \cdot \hat{p} + \beta[m + V_s(r)] + V_v(r)]\Psi(r) = E\Psi(r),
\]

where \( E \) is the energy of the particle, \( \hat{p} \) is the three-momentum operator. \( \alpha \), and \( \beta \) are the \( 4 \times 4 \) Dirac matrices written in terms of \( 2 \times 2 \) Pauli matrices and unit matrix. Under a spherical symmetry for which the potential fields depend on the radial coordinate, the quantum state of the particle is labelled by the quantum number set \((n_r, j, m, \kappa)\), where \( m \) is the projection of the total angular momentum on the \( z \)-axis and \( \kappa = \pm(j + 1/2) \) is the eigenvalues of the operator \( \hat{\kappa} = -\beta(\hat{\sigma} \cdot \hat{L} + 1) \) [73]. Here, \( \kappa = -(j + 1/2) \) denotes the aligned spin \((s_1/2, p_3/2, etc.)\), and \( \kappa = +(j + 1/2) \) denotes the unaligned spin \((p_1/2, d_3/2, etc.)\). The spherically symmetric Dirac wave function can then be written in terms of upper, and lower components as [74]

\[
\Psi(r) = \frac{1}{r} \left( \begin{array}{c} f(r)[Y_{\ell} \chi_{jm}]^j \vspace{1em} \\ ig(r)[Y_{\ell} \chi_{jm}]^j \end{array} \right),
\]

where \( f(r) \), and \( g(r) \) are the radial wave functions, \( Y_{\ell} (\theta, \phi) \) and \( \chi \) are the spherical, and spin functions, respectively. Substituting Eq (2) into Eq. (1), we get the following radial Dirac equations

\[
\left( \frac{d}{dr} + \frac{\kappa}{r} \right) f(r) - [M(r) + \epsilon]g(r) = 0,
\]

\[
\left( \frac{d}{dr} - \frac{\kappa}{r} \right) g(r) - [M(r) - \epsilon]f(r) = 0,
\]
with \( M(r) = m + V_s(r) \), and \( \epsilon = E - V_v(r) \). Using the expression for \( g(r) \) obtained from Eq. (3) and inserting it into Eq. (4), we have two second order differential equations including spin-orbit coupling term

\[
\left\{ \frac{d^2}{dr^2} - \frac{\kappa(\kappa+1)}{r^2} - [M^2(r) - \epsilon^2] \right\} f(r) = \left\{ \frac{1}{M(r) + \epsilon} \frac{d}{dr} [V_v(r) - V_s(r)] \left( \frac{d}{dr} + \frac{\kappa}{r} \right) \right\} f(r), \tag{5}
\]

\[
\left\{ \frac{d^2}{dr^2} - \frac{\kappa(\kappa-1)}{r^2} - [M^2(r) - \epsilon^2] \right\} g(r) = \left\{ \frac{1}{M(r) + \epsilon} \frac{d}{dr} [V_v(r) + V_s(r)] \left( \frac{d}{dr} - \frac{\kappa}{r} \right) \right\} g(r), \tag{6}
\]

Under the condition of the exact pseudo-spin symmetry, i.e., \( \frac{d}{dr} [V_v(r) + V_s(r)] = 0 \), or \( \Sigma(r) = V_v(r) + V_s(r) = const. \), Eq. (6) gives

\[
\left\{ \frac{d^2}{dr^2} - \frac{\kappa(\kappa-1)}{r^2} + [m - E + \Sigma][V_v(r) - V_s(r)] \right\} g(r) = [m^2 - E^2 + \Sigma(m + E)] g(r). \tag{7}
\]

From the last equation, the energy eigenvalues depend also on the quantum number \( \tilde{\ell} \) because of the relations given by \( \kappa(\kappa-1) = \tilde{\ell}((\tilde{\ell}+1) \), and \( \kappa(\kappa+1) = \ell(\ell+1) \). So, the energy eigenstates with \( j = \tilde{\ell} \pm 1/2 \) are degenerate for \( \tilde{\ell} \neq 0 \), which gives the situation of the exact pseudo-spin symmetry in the Dirac equation.

In the case of exact spin symmetry, i.e., \( \frac{d}{dr} [V_v(r) - V_s(r)] = 0 \), or \( \Delta(r) = V_v(r) - V_s(r) = const. \), Eq. (5) gives

\[
\left\{ \frac{d^2}{dr^2} - \frac{\kappa(\kappa+1)}{r^2} - [m + E - \Delta][V_v(r) + V_s(r)] \right\} f(r) = [m^2 - E^2 + \Delta(E - m)] f(r). \tag{8}
\]

III. THE PARAMETRIC NIKIFOROV-UVAROV APPROACH

A general form of a Schrödinger-like equation written for any potential can be parameterized as

\[
\left[ \frac{d^2}{ds^2} + \frac{\alpha_1 - \alpha_2 s}{s(1 - \alpha_3 s)} \frac{d}{ds} + \frac{-\xi_1 s^2 + \xi_2 s - \xi_3}{s(1 - \alpha_3 s)^2} \right] \psi(s) = 0, \tag{9}
\]

where the new parameters \( \alpha_i \), and \( \xi_i (i = 1, 2, 3) \) can be obtained by comparing Eq. (9) with the following ones
\[
\sigma^2(s) \frac{d^2 \psi(s)}{ds^2} + \sigma(s) \tilde{\tau}(s) \frac{d\psi(s)}{ds} + \tilde{\sigma}(s) \psi(s) = 0, \quad (10)
\]

The Eq. (10) is the standard form of the equation required in NU-method, where \(\sigma(s)\), \(\tilde{\sigma}(s)\) are polynomials at most second degree and \(\tilde{\tau}(s)\) is a first degree polynomial. So we obtain

\[
\tilde{\tau}(s) = \alpha_1 - \alpha_2 s; \quad \sigma(s) = s(1 - \alpha_3 s); \quad \tilde{\sigma}(s) = -\xi_1 s^2 + \xi_2 s - \xi_3, \quad (11)
\]

The NU-method are required the polynomial \(\pi(s)\), and a constant \(\lambda\) as \([35]\]

\[
\pi(s) = \frac{(\sigma' - \tilde{\tau})}{2} \pm \sqrt{\left(\frac{\sigma' - \tilde{\tau}}{2}\right)^2 - \tilde{\sigma} + k\sigma}, \quad (12)
\]

and

\[
\lambda = k + \pi'(s), \quad (13)
\]

The discriminant of the expression under the square root in the polynomial \(\pi(s)\) in Eq. (12) must be zero, which condition determines the constant \(k\). The polynomial \(\tau(s) = \tilde{\tau}(s) + 2\pi(s)\) is defined in terms of \(\pi(s)\) obtained by replacing \(k\) solved from the discriminant into Eq. (12), whose derivative must be negative \(\tau'(s) < 0\).

If \(\lambda\) in Eq. (13) satisfies

\[
\lambda = \lambda_n = -n\tau' - \frac{n(n-1)\sigma''}{2}, \quad n = 0, 1, 2, \ldots \quad (14)
\]

the hypergeometric type equation has a particular solution with degree \(n\), and one obtains the energy spectrum for the potential from Eq. (14).

The polynomial \(\pi(s)\) in Eq. (12) can be written in the parametric generalization approach by using Eq. (11) as

\[
\pi(s) = \alpha_4 + \alpha_5 s \pm \sqrt{(\alpha_6 - k\alpha_3)s^2 + (\alpha_7 + k)s + \alpha_8}, \quad (15)
\]
where $\alpha_4 = \frac{1}{2}(1 - \alpha_1); \alpha_5 = \frac{1}{2}(\alpha_2 - 2\alpha_3); \alpha_6 = \alpha_5^2 + \xi_1; \alpha_7 = 2\alpha_4\alpha_5 - \xi_2; \alpha_8 = \alpha_4^2 + \xi_3$.

In NU-method, the function under the square root in Eq. (15) must be the square of a polynomial, so we obtain

$$k_{1,2} = -\left(\alpha_7 + 2\alpha_3\alpha_8\right) \pm 2\sqrt{\alpha_8\alpha_9},$$

where $\alpha_9 = \alpha_3\alpha_7 + \alpha_3^2\alpha_8 + \alpha_6$.

For $k = -\left(\alpha_7 + 2\alpha_3\alpha_8\right) - 2\sqrt{\alpha_8\alpha_9}$ the polynomial $\pi(s)$ becomes

$$\pi(s) = \alpha_4 + \alpha_5 s - \left[\left(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}\right)s - \sqrt{\alpha_8}\right].$$

and also

$$\tau(s) = \alpha_1 + 2\alpha_4 - \left(\alpha_2 - 2\alpha_5\right)s - 2\left[\left(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}\right)s - \sqrt{\alpha_8}\right].$$

Thus, we obtain a negative value for the polynomial $\tau(s)$

$$\tau'(s) = -\left(\alpha_2 - 2\alpha_5\right) - 2\left(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}\right),$$

$$= -2\alpha_3 - 2\left(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}\right) < 0.$$  

From Eqs. (13), (17), (18), using $\tau(s) = \bar{\tau}(s) + 2\pi(s)$, and equating Eq. (13) with the condition that $\lambda$ should satisfy given by Eq. (14), we find the energy eigenvalue equation

$$\alpha_2 n - (2n + 1)\alpha_5 + (2n + 1)\left(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}\right) + n(n - 1)\alpha_3$$

$$+ \alpha_7 + 2\alpha_3\alpha_8 + 2\sqrt{\alpha_8\alpha_9} = 0.$$  

In the NU-method the general solution of Eq. (10) is suggested as a product of two independent parts [35]

$$\psi(s) = \phi(s) y(s),$$

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where $y(s)$ can be written by using the Rodriguez formula \[35\]

$$y_n(s) = \frac{B_n}{\rho(s)} \frac{d^n}{ds^n} \left[ \sigma^n(s) \rho(s) \right],$$

(22)

The weight function $\rho(s)$ should satisfy the condition

$$\frac{d}{ds} \left[ \sigma(s) \rho(s) \right] = \tau(s) \rho(s).$$

(23)

The other factor in the solution is defined as

$$\frac{\phi'(s)}{\phi(s)} = \frac{\pi(s)}{\sigma(s)}.$$\(\text{\small{\(24\)}}\)

To obtain the parametric generalization of the wave functions obtained in NU-method, we first use Eq. (23) to write the weight function $\rho(s)$

$$\rho(s) = s^{\alpha_{10}-1} (1 - \alpha_3 s)^{\frac{\alpha_{11}}{\alpha_3} - \alpha_{10}-1},$$

(25)

which gives by using Eq. (22)

$$y_n(s) = P_n^{(\alpha_{10}-1, \frac{\alpha_{11}}{\alpha_3} - \alpha_{10}-1)}(1 - 2\alpha_3 s),$$

(26)

where $\alpha_{10} = \alpha_1 + 2\alpha_4 + 2\sqrt{\alpha_8}$; $\alpha_{11} = \alpha_2 - 2\alpha_5 + 2(\sqrt{\alpha_9} + \alpha_3 \sqrt{\alpha_8})$, and $P_n^{(\alpha, \beta)}(1 - 2\alpha_3 s)$ are Jacobi polynomials. From Eq. (24), one gets

$$\phi(s) = s^{\alpha_{12}} (1 - \alpha_3 s)^{-\alpha_{12} - \frac{\alpha_{13}}{\alpha_3}},$$

(27)

Finally, the general solution is written in terms of parameters

$$\psi(s) = \phi(s)y(s),$$

(28)

$$\psi(s) = s^{\alpha_{12}} (1 - \alpha_3 s)^{-\alpha_{12} - \frac{\alpha_{13}}{\alpha_3}} P_n^{(\alpha_{10}-1, \frac{\alpha_{11}}{\alpha_3} - \alpha_{10}-1)}(1 - 2\alpha_3 s).$$

(29)
where $\alpha_{12} = \alpha_4 + \sqrt{\alpha_8}$; $\alpha_{13} = \alpha_5 - (\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8})$.

In some problems the situation appears where $\alpha_3 = 0$. For this type of the problems when

$$
\lim_{\alpha_3 \rightarrow 0} P_n^{(\alpha_{10} - 1, \frac{\alpha_{11}}{\alpha_3} - \alpha_{10} - 1)} (1 - \alpha_3 s) = L_n^{\alpha_{10} - 1}(\alpha_{11} s),
$$

and

$$
\lim_{\alpha_3 \rightarrow 0} (1 - \alpha_3 s)^{-\alpha_{12} - \frac{\alpha_{14}}{\alpha_3}} = e^{\alpha_{13} s},
$$

the solution given in Eq. (29) becomes as

$$
\psi(s) = s^{\alpha_{12}} e^{\alpha_{13} s} L_n^{\alpha_{10} - 1}(\alpha_{11} s),
$$

and the energy spectrum is

$$
\alpha_2 n - 2\alpha_5 n + (2n + 1)(\sqrt{\alpha_9} - \alpha_3\sqrt{\alpha_8}) + n(n - 1)\alpha_3 + \alpha_7 \\
+ 2\alpha_3\alpha_8 - 2\sqrt{\alpha_8\alpha_9} + \alpha_5 = 0.
$$

### IV. ENERGY SPECTRA AND WAVE FUNCTIONS

#### A. Bound States of the Morse Potential

We assume that the ”sum” potential $\Sigma(r) = V_v(r) + V_s(r)$ is the $PT$-symmetric generalized Morse potential [75] given by

$$
V(r) = V_1 e^{-2\alpha r} - V_2 e^{-i\alpha r},
$$

where the constants $V_1$ and $V_2$ are related with the dissociation energy, and $\alpha$ is the potential width.

Substituting Eq. (34) into Eq. (7), and by using the coordinate transformation $s = e^{-i\alpha r}$, we get the following equation in the exact pseudo-spin symmetry for $\kappa = 0$.
\[
\left\{ \frac{d^2}{ds^2} + \frac{1}{s} \frac{d}{ds} - \frac{1}{s^2} \left[ 4\beta^2 \left( \mu^2 - E^2 + E(\mu + \Sigma) \right) + 4\beta^2 V_2(\mu - E + \Sigma)s + 4\beta^2 V_1(E - \mu - \Sigma)s^2 \right] \right\} g(s) = 0. \tag{35}
\]

Comparing Eq. (35) with Eq. (9), we obtain the following parameter set

\[
\begin{align*}
\alpha_1 &= 1, & \xi_1 &= 4\beta^2 V_1(\mu - \Sigma) \\
\alpha_2 &= 0, & \xi_2 &= 4\beta^2 V_2(\mu - \Sigma) \\
\alpha_3 &= 0, & \xi_3 &= 4\beta^2 (E^2 - \mu^2 - E(\mu + \Sigma)) \\
\alpha_4 &= 0, & \alpha_5 &= 0 \\
\alpha_6 &= \xi_1, & \alpha_7 &= -\xi_2 \\
\alpha_8 &= \xi_3, & \alpha_9 &= \xi_1 \\
\alpha_{10} &= 1 + 2\sqrt{\xi_3}, & \alpha_{11} &= 2\sqrt{\xi_1} \\
\alpha_{12} &= \sqrt{\xi_3}, & \alpha_{13} &= -\sqrt{\xi_1}
\end{align*}
\tag{36}
\]

From Eq. (33), we obtain the energy eigenvalue equation of \emph{PT}-symmetric, generalized Morse potential under the exact pseudospin symmetry for \(\kappa = 0\)

\[
E^2 - E(\mu + \Sigma) - \mu^2 = \frac{1}{16\beta^2} \left( 2n + 1 - \frac{2\beta V_2}{\sqrt{V_1}} \sqrt{E - \mu - \Sigma} \right)^2, \tag{37}
\]

where \(\beta^2 = 1/4\alpha^2\). It can be seen from Eq. (37) that the energy eigenvalue equation has a quadratic form which gives positive and also negative energy solutions. However, the negative energy states exist only in the exact pseudo-spin limit. Thus we should choose the negative energy solution in Eq. (37).

The corresponding lower, and upper spinor components can be obtained from Eq. (32), and Eq. (4), respectively,

\[
g(s) = s^{2\beta} \sqrt{E^2 - \mu^2 - E(\mu + \Sigma)} e^{-2\beta \sqrt{V_1(E - \mu - \Sigma)}} s
\]  
\[
\times \left( \frac{4\beta}{\sqrt{V_1}} \sqrt{E^2 - \mu^2 - E(\mu + \Sigma)} \right) ^{4\beta \sqrt{V_1(E - \mu - \Sigma)}} s, \tag{38}
\]

and
\[ f(s) = i(\mu + \Sigma - E)s^{2\beta}e^{2\beta\sqrt{E^2 - \mu^2 - E(\mu + \Sigma)}}e^{-2\beta\sqrt{V_1(E - \mu - \Sigma)}s} \]
\[ \times \left\{ \left[ \sqrt{V_1(E - \mu - \Sigma)} s - \sqrt{E^2 - \mu^2 - E(\mu + \Sigma)} \right] \right. \]
\[ \times \left. L_n^{(4\beta\sqrt{E^2 - \mu^2 - E(\mu + \Sigma)})} \left( 4\beta\sqrt{V_1(E - \mu - \Sigma)} s \right) \right. \]
\[ + 2\sqrt{V_1(E - \mu - \Sigma)} sL_n^{(1+4\beta\sqrt{E^2 - \mu^2 - E(\mu + \Sigma)})} \left( 4\beta\sqrt{V_1(E - \mu - \Sigma)} s \right) \right\}. \tag{39} \]

where \( L_n^{(k)}(x) \) are the Laguerre polynomials, and we have used some recursion relations, which is \( \frac{d}{dx}L_n^{(k)}(x) = -L_n^{(1+k)}(x) \), to obtain the upper component given in Eq. (39).

Now, we solve the Dirac equation in the case of exact spin symmetry, where the "difference" potential \( \Delta(r) = V_v(r) - V_s(r) \) is a constant, let say \( \Delta(r) = \Delta = \text{const.} \). In this case we set the "sum" potential \( \Sigma(r) \) as the Morse potential in Eq. (34). Substituting the potential into Eq. (8), and using the same variable \( s = e^{-i\alpha r} \), we obtain

\[ \left\{ \frac{d^2}{ds^2} + \frac{1}{s} \frac{d}{ds} + \frac{1}{s^2} \left[ 4\beta^2 \left( \mu^2 - E^2 + \Delta(E - \mu) \right) \right. \]
\[ + \left. 4\beta^2 V_2(-\mu - E + \Delta)s + 4\beta^2 V_1(E + \mu - \Delta)s^2 \right] \right\} g(s) = 0, \tag{40} \]

Comparing Eq. (40) with Eq. (9), we obtain the following parameter set

\[ \alpha_1 = 1, \quad \xi_1 = 4\beta^2 V_1(\Delta - E - \mu) \]
\[ \alpha_2 = 0, \quad \xi_2 = 4\beta^2 V_2(\Delta - E - \mu) \]
\[ \alpha_3 = 0, \quad \xi_3 = 4\beta^2(\mu^2 - E^2 + \Delta(E - \mu)) \]
\[ \alpha_4 = 0, \quad \alpha_5 = 0 \]
\[ \alpha_6 = \xi_1, \quad \alpha_7 = -\xi_2 \]
\[ \alpha_8 = \xi_3, \quad \alpha_9 = \xi_1 \]
\[ \alpha_{10} = 1 + 2\sqrt{\xi_3}, \quad \alpha_{11} = 2\sqrt{\xi_1} \]
\[ \alpha_{12} = \sqrt{\xi_3}, \quad \alpha_{13} = -\sqrt{\xi_1} \tag{41} \]

From Eq. (33), we obtain the energy eigenvalue equation in the case of exact spin symmetry for \( \kappa = 0 \)

\[ E^2 - \mu^2 + \Delta(\mu - E) = \frac{1}{16\beta^2} \left( 2n + 1 - \frac{2\beta V_2}{\sqrt{V_1}} \sqrt{\Delta - E - \mu} \right)^2. \tag{42} \]
where $\beta^2 = 1/4\alpha^2$. It can be seen from Eq. (42) that the energy eigenvalue equation has a quadratic form, which gives positive and also negative energy solutions. But the positive energy levels exist only in the exact spin limit. So we should choose the positive energy solution in Eq. (42).

The corresponding Dirac spinor can be written as

$$f(s) = s^{2\beta} \sqrt{E^2 - \mu^2 + \Delta(E - \mu)} e^{-2\beta \sqrt{V_1} (\Delta - E - \mu)} s \times \left(4\beta V_1 \left(\frac{\Delta - E - \mu}{\mu + E} + 4V_0\right) s\right).$$

(43)

**B. Bound States of the Pöschl-Teller Potential**

In this section, we search the bound states solutions and the corresponding Dirac spinors of the $PT$-symmetric, $q$-deformed Pöschl-Teller potential [76] given as

$$V(r) = -4V_0 \frac{e^{-2i\alpha r}}{(1 + qe^{-2i\alpha r})^2},$$

(44)

where $V_0$ and $\alpha$ are constant potential parameters, and the parameter $q$ describes the deformation in potential having the value $0 < q < 1$.

In the case of exact pseudo-spin symmetry, we assume that the ”difference” potential $\Delta(r)$ is the Pöschl-Teller potential. Substituting Eq. (44) into Eq. (7), by using the new coordinate transformation $s = -e^{-2i\alpha r}$, and taking into account $\Sigma(r) = \Sigma = const.$, we obtain

$$\frac{d^2 g(s)}{ds^2} + \frac{1 - qs}{s(1 - qs)} \frac{dg(s)}{ds} + \frac{1}{[s(1 - qs)]^2} \left\{\beta^2 (E - \mu - \Sigma) \left[\mu + E - [2q(\mu + E) + 4V_0]s + q^2 (\mu + E)s^2\right]\right\} g(s) = 0.$$ 

(45)

Following the same procedure in the last section, we obtain the parameter set
\[\begin{align*}
\alpha_1 &= 1, & \xi_1 &= \beta^2 q^2 (\mu + E)(\mu + \Sigma - E) \\
\alpha_2 &= q, & \xi_2 &= 2 \beta^2 (\mu + \Sigma - E)[q(\mu + E) + 2V_0] \\
\alpha_3 &= q, & \xi_3 &= \beta^2 (\mu + E)(\mu + \Sigma - E) \\
\alpha_4 &= 0, & \alpha_5 &= -\frac{q}{2} \\
\alpha_6 &= \xi_1 + \frac{q^2}{4}, & \alpha_7 &= -\xi_2 \\
\alpha_8 &= \xi_3, & \alpha_9 &= \xi_1 - q\xi_2 + q^2 \xi_3 + \frac{q^2}{4} \\
\alpha_{10} &= 1 + 2\sqrt{\xi_3}, & \alpha_{11} &= 2q + 2(\sqrt{\xi_1 - q\xi_2 + q^2 \xi_3 + \frac{q^2}{4}} + q\sqrt{\xi_3}) \\
\alpha_{12} &= \sqrt{\xi_3}, & \alpha_{13} &= -\frac{q}{2} - (\sqrt{\xi_1 - q\xi_2 + q^2 \xi_3 + \frac{q^2}{4}} + q\sqrt{\xi_3}) \\
\end{align*}\]

and the energy eigenvalue equation of \(PT\)-symmetric Pöschl-Teller potential under the exact pseudospin symmetry for \(\kappa = 0\) from Eq. \((33)\)

\[
\begin{align*}
\mu + \Sigma - E - (2n + 1)\frac{q}{4V_0\beta} \sqrt{(\mu + E)(\mu + \Sigma - E)} \\
- \sqrt{\frac{q^2}{16V_0^2\beta^4}} - \frac{q}{V_0\beta}(\mu + \Sigma - E) \left[n + \frac{1}{2} + \beta \sqrt{(\mu + E)(\mu + \Sigma - E)} \right] \\
- \frac{q}{4V_0\beta^2}[n(n + 1) + \frac{1}{2}] = 0.
\end{align*}\]  

The last energy eigenvalue equation has a quadratic form in terms of energy \(E\), so we obtain positive and also negative energy eigenvalues, but we should take the negative energy values in the exact pseudospin limit.

The corresponding Dirac lower and upper spinors can be written in terms of Jacobi polynomials, i.e., \(P_n^{(\alpha, \beta)}(x)\), by using Eq. \((29)\) as

\[
\begin{align*}
g(s) &= s^{\beta} \sqrt{(\mu + E)(\mu + \Sigma - E)} \left(1 - qs\right)^{\frac{1}{2}} \left[1 + \sqrt{1 - \frac{16V_0\beta^2}{q}(\mu + \Sigma - E)} \right] \\
&\times P_n^{(2\beta)\sqrt{(\mu + E)(\mu + \Sigma - E)}, \sqrt{1 - \frac{16V_0\beta^2}{q}(\mu + \Sigma - E)}} \left(1 - 2qs\right), \\
\end{align*}\]  

and

\[
\begin{align*}
f(s) &= i s^{1 + \beta} \sqrt{(\mu + E)(\mu + \Sigma - E)} \left(1 - qs\right)^{\frac{1}{2}} \left[1 + \sqrt{1 - \frac{16V_0\beta^2}{q}(\mu + \Sigma - E)} \right] \\
&\times \beta(\mu + \Sigma - E).
\end{align*}\]
\[
\times \left\{ \frac{1}{s} \left[ \beta \sqrt{(\mu + E)(\mu + \Sigma - E)} - \frac{q}{2} \left( 1 + \sqrt{1 - \frac{16V_0\beta^2}{q}(\mu + \Sigma - E)} \right) \right] s \right\} \]

\[
P_n(2\beta \sqrt{(\mu + E)(\mu + \Sigma - E)}, \sqrt{1 - \frac{16V_0\beta^2}{q}(\mu + \Sigma - E)}) (1 - 2qs)
\]

\[
- q \left( n + 2\beta \sqrt{(\mu + E)(\mu + \Sigma - E)} + \sqrt{1 - \frac{16V_0\beta^2}{q}(\mu + \Sigma - E) + 1} \right)
\]

\[
P_{n-1}(1 + 2\beta \sqrt{(\mu + E)(\mu + \Sigma - E)}, 1 + \sqrt{1 - \frac{16V_0\beta^2}{q}(\mu + \Sigma - E)}) (1 - 2qs) \right\}.
\]

(49)

They are obtained by using some recursion relation of Jacobi polynomials, i.e.,
\[
\frac{d}{ds} P_n^{(\alpha, \beta)}(x) = \frac{1}{2}(n + \alpha + \beta + 1) P_{n-1}^{(1+\alpha, 1+\beta)}(x).
\]

As a last case, we obtain the solutions of the Dirac equation in the case of exact spin symmetry for the \PT-symmetric Pöschl-Teller potential. In this case the "difference" potential is a constant, i.e., \[\Delta(r) = \Delta = \text{const.}\], and we set the "sum" potential as \PT-symmetric Pöschl-Teller potential given in Eq. (44). We obtain from Eq. (8) by using the coordinate transformation \[s = -e^{-2i\alpha r}\]

\[
\frac{d^2 f(s)}{ds^2} + \frac{1 - qs}{s(1 - qs)} \frac{df(s)}{ds} + \frac{\mu + E - \Delta}{|s(1 - qs)|^2} \left\{ \beta^2 (\mu - E) + \beta^2 [2q(E - \mu) + 4V_0] s + \beta^2 q^2 (\mu - E)s^2 \right\} f(s) = 0,
\]

(50)

Following the same procedure in the last section, we obtain the parameter set

\[
\alpha_1 = 1,
\xi_1 = \beta^2 q^2 (\mu - E)(\mu + E - \Delta)
\]

\[
\alpha_2 = q,
\xi_2 = 2\beta^2 (\mu + E - \Delta)[q(E - \mu) + 2V_0]
\]

\[
\alpha_3 = q,
\xi_3 = \beta^2 (E - \mu)(\mu + E - \Delta)
\]

\[
\alpha_4 = 0,
\alpha_5 = -\frac{q}{2}
\]

\[
\alpha_6 = \xi_1 + \frac{q^2}{4},
\alpha_7 = -\xi_2
\]

\[
\alpha_8 = \xi_3,
\alpha_9 = \xi_1 - q\xi_2 + q^2\xi_3 + \frac{q^2}{1}
\]

\[
\alpha_{10} = 1 + 2\sqrt{\xi_3},
\alpha_{11} = 2q + 2(\sqrt{\xi_1 - q\xi_2 + q^2\xi_3 + \frac{q^2}{4}} + q\sqrt{\xi_3})
\]

\[
\alpha_{12} = \sqrt{\xi_3},
\alpha_{13} = -\frac{q}{2} - (\sqrt{\xi_1 - q\xi_2 + q^2\xi_3 + \frac{q^2}{4}} + q\sqrt{\xi_3})
\]

(51)

The energy eigenvalue equation of \PT-symmetric Pöschl-Teller potential under the exact spin symmetry for \[\kappa = 0\] from Eq. (33) is obtained
\[
\sqrt{1 - \frac{16\beta^2 V_0}{q}} (\mu + E - \Delta) \left[ n + \frac{1}{2} + \beta \sqrt{(E - \mu)(\mu + E - \Delta)} \right] \\
+ \sqrt{\mu + E - \Delta} \left[ (2n + 1)\beta \sqrt{E - \mu} - \frac{4\beta^2 V_0}{q} \sqrt{\mu + E - \Delta} \right] + n(n + 1) + \frac{1}{2} = 0. \tag{52}
\]

Finally, we obtain the corresponding Dirac spinor from Eq. (29)

\[
f(s) = s^3 \sqrt{(E - \mu)(\mu + E - \Delta)} (1 - qs)^{\frac{1}{2}} \left[ 1 + \sqrt{1 - \frac{16V_0\beta^2}{q}(\mu + E - \Delta)} \right] \\
\times P_n^{(2\beta \sqrt{(E\mu)(\mu + E - \Delta)}, \sqrt{1 - \frac{16V_0\beta^2}{q}(\mu + E - \Delta)})(1 - 2qs). \tag{53}
\]

V. CONCLUSION

We have studied the energy spectra, and corresponding lower, and upper spinor components of the Dirac equation with exact pseudospin, and spin symmetry for \(PT\)-symmetric generalized Morse and Pöschl-Teller potentials. We have obtained the results by using the parametric generalization of the Nikiforov-Uvarov method. We have presented the energy eigenvalue equations for two \(PT\)-symmetric potentials, and obtained the corresponding Dirac spinors in terms of Laguerre (Jacobi) polynomials for the Morse (Pöschl-Teller) potential for the value of spin-orbit quantum number \(\kappa = 0\).

VI. ACKNOWLEDGMENTS

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