Analytic Solution of Heat Conduction Problem Involving Moving Interface With Spherical Boundaries

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A solid is bounded by two concentric spherical surfaces and the whole system is at a temperature T_A which is below the freezing point T_f of the substance. At a certain instant, one of the surfaces is raised to and maintained at a temperature T_B which is above the freezing point T_f, while the other surface is maintained at the original temperature T_A. The problem of finding the temperature distribution in the solid and the liquid phases and the position of the interface between the two phases and the rate of motion of the interface are solved by an analytic method in which a solution of the heat conduction problem is constructed by the superposition of solutions such that all the initial, asymptotic and boundary conditions of the problem are satisfied.

1. STATEMENT OF THE PROBLEM

THE problem of this article is the same in principle as that of the article entitled "Analytic Solution of Heat Conducction with Phase Change" [1]. The differece in the two problems is in the geometrical boundaries. In the previous problem the system is bounded by two parallel plates and in the present problem it is bounded by two concentric spherical surfaces.

A substance is bounded by two fixed concentric spherical surfaces A and ***B*** of radius r_A and r_B, respectively. The whole system is at temperature T_A which is below the freezing point T_f of the substance. At a certain instant the surface ***B*** is heated to T_B and maintained at T_B which is above T_f. The temperature of surface ***A*** is maintained at T_A. The solid near the surface ***B*** begins to melt and the interface between the solid and liquid moves from the surface ***B*** toward surface ***A*** until a steady state is reached. The temperature distribution as a function of time and the distance from the center of the spherical surfaces depends on the temperature T_A, T_B, and T_f, the radii of the spherical surfaces, the heat conductivity, density, specific heat of the solid and liquid phases and the

(1) Kuan Shih Yuan Wu, *Analytic Solution of Heat Conduction Problem with Phase Change*, Chinese J. of Phys. 6, 29 (1968).

heat of fusion of the substance. From the temperature distribution, the position of the interface at any time and its rate of motion are determined.

It is assumed that the physical properties of the substance are constant, independent of temperature, and that the heat transport by convection and the volume change on melting ane negligible. The problem is then one of the form[(2)]

$$\frac{\partial T}{\partial t}=a\left(\frac{\partial^2 T}{\partial r^2}+\frac{2}{r}\frac{\partial T}{\partial r}\right)$$

where a is the thermal diffusivity, $T(r,t)$ the temperature, t the time and r the coordinate of a point with the center of the spherical surfaces as the origin of the spherical coordinate system.

2. SOLUTION OF THE PROBLEM

Let r_A, r_B be the radius of the inner and outer concentric spherical surfaces A and B respectively with origin at 0, r_1, r_2 the coordinates of a point in the phase 1, say solid phase and in the phase 2, the liquid phase respectively, r_f the radius of the spherical interface, $T_1(r_1,t), T_2(r_2,t)$ the temperature at points on the spherical surfaces of radii, r_1 and r_2 at time t as shown in Figure 1.

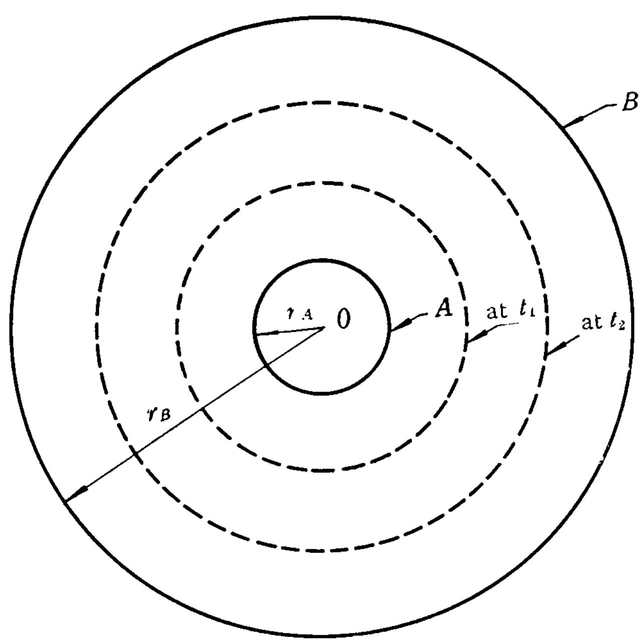

Fig. 1.

(2) H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids*, 2nd Edition, p. 230 (D. R. Hil!man and Sons, Ltd., England, 1962).

The differential equations of conduction for phase 1 and phase 2 take the following forms.

For phase 1

$$\frac{\partial T_1}{\partial t}=a_1\left(\frac{\partial^2 T_1}{\partial r_1^2}+\frac{2}{r_1}\frac{\partial T_1}{\partial r_1}\right),\qquad a_1=\frac{k_1}{\rho_1 c_{v1}}. \tag{1}$$

For phase 2

$$\frac{\partial T_2}{\partial t}=a_2\left(\frac{\partial^2 T_2}{\partial r_2^2}+\frac{2}{r_2}\frac{\partial T_2}{\partial r_2}\right),\qquad a_2=\frac{k_2}{\rho_2 c_{v2}}, \tag{2}$$

where a_1, k_1, ρ_1, c_{v1} and $\rho_2, k_2, \rho_2, c_{v2}$ are the thermal diffusivity, the thermal conductivity, the density and the specific heat at constant volume of phase 1 and phase 2 respectively.

To express the conduction equations in dimensionless form, the following dimensionless variables are introduced.

Let

$$\tau_1=(T_1-T_A)/(T_B-T_A),$$

$$\tau_2=(T_2-T_A)/(T_B-T_A),$$

$$\tau_f=(T_f-T_A)/(T_B-T_A),$$

$$x_A=r_A/(r_B-r_A),\ x_B=r_B/(r_B-r_A),$$

$$x_1=r_1/(r_B-r_A),\ x_2=r_2/(r_B-r_A),$$

$$X=r_f/(r_B-r_A),\ X_\infty=r_f/(r_B-r_A)\Big|_{t\to\infty},$$

$$\theta=a_1 t/(r_B-r_A)^2.$$

Equation (1) and (2) with substitution of the dimensionless variables as defined above become :

$$\frac{\partial\tau_1}{\partial\theta}=\frac{\partial^2\tau_1}{\partial x_1^2}+\frac{2}{x_1}\frac{\partial\tau_1}{\partial x_1}, \tag{1'}$$

$$\frac{\partial\tau_2}{\partial\theta}=\beta\left(\frac{\partial^2\tau_2}{\partial x_2^2}+\frac{2}{x_2}\frac{\partial\tau_2}{\partial x_2}\right) \tag{2'}$$

where $\beta=a_2/a_1$.

Let

$$\tau_1=\Theta(\theta)R(x_1)/x_1.$$

Then

$$\frac{1}{\Theta}\frac{d\Theta}{d\theta}=\frac{1}{R}\frac{d^2R}{dx_1^2}=-b^2 \tag{2''}$$

where b^2 is a constant. The solutions of these two equations are

$$\Theta(\theta)=e^{-b^2\theta}$$

$$R(x_1)=\sin b(x_1+c)$$

where c is a constant.

One special solution of an eqation of the form (1′) is

$$\tau_1=A+a_0/x_1, \qquad A,\ a_0=\text{constant},$$

which corresponds to the case $b=0$ in (2″) above. Another special solution is of the type of $\frac{1}{x_1}\Phi\left(\frac{x_1}{2\sqrt{\theta}}\right)$ where Φ is an error function

$$\Phi(y)=\frac{2}{\sqrt{\pi}}\int_0^y e^{-y^2}dy.$$

We shall construct solutions of (1′) and (2′) as follows:

$$\tau_1=A+\frac{a_0}{x_1}+\frac{1}{x_1}\sum_{n=1} a_n e^{-b_n^2\theta}\sin b_n(x_1+c)+\frac{p}{x_1}\Phi\left(\frac{x_1-x_A}{2\sqrt{\theta}}\right), \tag{3}$$

$$\tau_2=B+\frac{f_0}{x_2}+\frac{1}{x_2}\sum_{n=1} f_n e^{-\beta g_n^2\theta}\sin g_n(x_2+d)+\frac{h}{x_2}\Phi\left(\frac{1-x_2+x_A}{2\sqrt{\beta\theta}}\right), \tag{4}$$

where A, a_0, a_n, b_n, c, P, B, f_0, f_n, g_n, d, h, are constants.

The initial, boundary and asymtotic conditions to be satisfied for the solutions are the following:

Initial conditions at $\theta=0$:

$$\tau_1=0 \quad \text{for} \quad x_A\leq x_1<x_B \tag{5}$$

$$\tau_2=1 \quad \text{at} \quad x_2=x_B. \tag{6}$$

The steady state is given by the asymtotic condition on $B\to\infty$:

$$\tau_1=0 \quad \text{at} \quad x_1=x_A \tag{7}$$

$$\tau_1=\tau_f \quad \text{at} \quad x_1=X_\infty \tag{8}$$

$$\tau_2=\tau_f \quad \text{at} \quad x_2=X_\infty, \tag{9}$$

$$\tau_2=1 \quad \text{at} \quad x_2=x_B. \tag{10}$$

At the interface in the steady state, we have

$$\frac{\partial\tau_1}{\partial x_1}\bigg|_{X_\infty}-r\frac{\partial\tau_2}{\partial x_2}\bigg|_{X_\infty}=0,\quad r=k_2/k_1. \tag{11}$$

Boundary condition :

$$\tau_1=0 \quad \text{at} \quad x_1=x_A, \tag{12}$$

$$\tau_2=1 \quad \text{at} \quad x_2=x_B. \tag{13}$$

To satisfy these, we choose

$$c=-x_A,\quad d=-x_A, \tag{14}$$

with

$$b_n=n\pi,\quad g_n=n\pi.$$

Substitution of the initial, boundary and asymtotic conditions given above in the solutions equations (3) and (4) gives the following set of equations

$$0=A+\frac{a_0}{x_1}+\frac{1}{x_1}\sum_{n=1} a_n \sin n\pi(x_1-x_A)+\frac{p}{x_1}, \tag{5'}$$

$$B=1-\frac{f_0}{x_B}, \tag{6'}$$

$$0=A+\frac{a_0}{x_A}, \tag{7'}$$

$$\tau_f=A+\frac{a_0}{X_\infty}, \tag{8'}$$

$$\tau_f=B+\frac{f_0}{X_\infty}, \tag{9'}$$

$$1=B+\frac{f_0}{x_B}, \tag{10'}$$

$$f_0=\frac{a_0}{r}. \tag{11'}$$

From these equations (6') to (11'), the following constants are determined.

$$a_0=-x_A x_B[\tau_f+r(1-\tau_f)]/(x_B-x_A), \tag{14-l}$$

$$A=x_B[\tau_f+r(1-\tau_f)]/(x_B-x_A), \tag{14-Z}$$

$$B=[r x_B+(1-r)\tau_f x_A]/r(x_B-x_A), \tag{14-3}$$

$$X_\infty=x_B[\tau_f+r(1-\tau_f)]/[\tau_f+r(1-\tau_f)\frac{x_B}{x_A}]. \tag{14-4}$$

Hence finally

$$\tau_1=A+\frac{a_0}{x_1}+\frac{1}{x_1}\sum_{n=1} a_n e^{-n^2\pi^2\theta}\ \sin n\pi(x_1-x_A)+\frac{p}{x_1}\ \Phi\left(\frac{x_1-x_A}{2\sqrt{\theta}}\right), \tag{15}$$

$$\tau_2=B+\frac{f_0}{x_2}+\frac{1}{x_2}\sum_{n=1} f_n e^{-\beta n^2\pi^2\theta}\ \sin n\pi(x_2-x_A)+\frac{h}{x_2}\ \Phi\left(\frac{1-x_2+x_A}{2\sqrt{\beta\theta}}\right). \tag{16}$$

The a,, are to be determined by (5') in terms of a_0 and P,

$$\frac{n\pi}{2}a_n=-\frac{1}{x_A}(-1)^n a_0-[1-(-1)^n]p,\qquad n\geq 1. \tag{17}$$

Another condition is the heat balance equation for the melting of the solid

$$k_1 \frac{\partial T_1}{\partial r_1}\bigg|_{r_f} - k_2 \frac{\partial T_2}{\partial r_2}\bigg|_{r_f} = \rho Q \frac{\partial r_f}{\partial t}$$

where Q is the heat of fusion of the substance, or in dimensionless form,

$$\frac{\partial \tau_1}{\partial x_1}\bigg|_X - \gamma \frac{\partial \tau_2}{\partial x_2}\bigg|_X = q \frac{\partial X}{\partial \theta}, \tag{18}$$

where $\gamma = k_2/k_1$ and $q = \rho Q / c_{v1}(T_B - T_A)$.

The rate of motion of the interface is given by the equation

$$\frac{\partial X}{\partial \theta}\bigg|_{\text{from phase 1}} = \frac{\partial X}{\partial \theta}\bigg|_{\text{from phase 2}} \tag{19}$$

Now from (15),(16), we have for the interface

$$\tau_f = -\frac{a_0}{x_A} + \frac{a_0}{X} + \frac{1}{X}\sum_{n=1} a_n e^{-n^2\pi^2\theta} \sin n\pi(X - x_A) + \frac{p}{X}\Phi\left(\frac{X - x_A}{2\sqrt{\theta}}\right), \tag{20}$$

$$\tau_f = 1 + \frac{a_0}{\gamma}\left(\frac{1}{X} - \frac{1}{x_B}\right) + \frac{1}{X}\sum_{n=1} f_n e^{-\rho n^2\pi^2\theta} \sin n\pi(X - x_A) + \frac{h}{X}\Phi\left(\frac{1 - X + x_A}{2\sqrt{\beta\theta}}\right). \tag{21}$$

Let (20) and (21) be represented as

$$F(X, \theta) = 0 \quad \text{and} \quad G(X, \theta) = 0.$$

Then

$$\frac{\partial X}{\partial \theta}\bigg|_{\text{from phase 1}} = -\frac{\left(\frac{\partial F}{\partial \theta}\right)_X}{\left(\frac{\partial F}{\partial X}\right)_\theta}, \tag{22-1}$$

$$\frac{\partial X}{\partial \theta}\bigg|_{\text{from phase 2}} = -\frac{\left(\frac{\partial G}{\partial \theta}\right)_X}{\left(\frac{\partial G}{\partial X}\right)_\theta}. \tag{22-2}$$

Let X_i be the position of the interface at a time θ_i, and let us introduce the following notations :

$$\Psi_1(i) = \frac{1}{\sqrt{\pi\theta_i}} \exp\left[-\frac{(X_i - x_A)^2}{4\theta_i}\right], \tag{23-1}$$

$$\Psi_2(i)\equiv\frac{1}{\sqrt{\beta\pi\theta_i}}\exp\left[-\frac{(1-X_i+x_A)^2}{4\beta\theta_i}\right], \tag{23-Z}$$

$$B(i)\equiv\sum_{n=1}n^2\pi^2a_n\exp(-n^2\pi^2\theta_i)\sin n\pi(X_i-x_A), \tag{23-3}$$

$$C(i)\equiv\sum_{n=1}n\pi a_n\exp(-n^2\pi^2\theta_i)\cos n\pi(X_i-x_A), \tag{23-4}$$

$$D_n(i)\equiv n^2\pi^2\exp(-\beta n^2\pi^2\theta_i)\sin n\pi(X_i-x_A), \tag{23-5}$$

$$E(i)\equiv n\pi\exp(-\beta n^2\pi^2\theta_i)\cos n\pi(X_i-x_A), \tag{23-6}$$

$$M(i)\equiv-\frac{B(i)+p(X_i-x_A)\Psi_1(i)/2\theta_i}{-\tau_f-\dfrac{a_0}{x_A}+C(i)+p\Psi_1(i)}, \tag{23-7}$$

$$N(i)\equiv-\frac{\beta\sum_n D_n(i)f_n+h(1-X_i+x_A)\Psi_2(i)/2\theta_i}{-\tau_f+1-\dfrac{a_0}{rx_B}+\sum_n E_n(i)f_n-h\Psi_2(i)}. \tag{23-8}$$

Equation (19) then becomes

$$M(i)=N(i), \tag{19-A}$$

and equation (18) becomes, by use of (14-1),

$$C(i)+p\Psi_1(i)-r[\sum_n E_n(i)f_n-h\Psi_2(i)]=-qM(i). \tag{18-A}$$

In equations (19-A),(18-A), the as yet undetermined constants are P, h, and the f_n's. To determine them, the following procedure is suggested.

Let us take equation (20) in which the expression (17) for a,, in terms of a_0 and p is used:

$$\tau_f=-\frac{a_0}{x_A}+\frac{a_0}{X}-\frac{2}{\pi X}\sum_n\left[\frac{(-1)^n}{x_A}a_0+[1-(-1)^n]p\right]\frac{1}{n}\exp(-n^2\pi^2\theta)\times$$
$$\sin n\pi(X-x_A)+\frac{p}{X}\Phi\left(\frac{X-x_A}{\sqrt{2\theta}}\right). \tag{24}$$

For any assumed value of P, this equation give the position of the interface X as a function of time θ. Let us start by assuming a value $p=p_\mu$, and with it, find m pairs of value $\{X_i(\mu),\theta_i(\mu)\}, i=1,2,\ldots,m$. On putting each pair in equation (21) in turn, we get for the m pairs m equations in which h and the f_n are unknowns. Let the series be truncated at the $(2m-2)$-th term.

Let us next calculate M(i) from (23-7) for each pair {X(/l), $\theta_i(\mu)\}$ for $p=p_\mu$. Also C(i), $\Psi_1(i)$, E,(i), $\Psi_2(i)$ in (18-A) are calculated for $\{X_i(\mu),\theta_i(\mu)\}, i=1,2,\ldots,m$. From (18-A) we have then m equations in which P, h on the left hand side and the f_n are unknowns. Also truncate the series in (18-A) at the $(2m-2)$-th term.

Equations (21) and (18-A) then give $2m$ linear equations in which the unknowns are

$$p, h, f_1, f_2, \ . \ . \ . \ f_{2m-2}.$$

Solution of this system will give a

$$p(\mu),\ h(\mu),\ f_1(\mu), \ldots f_{2m-2}(\mu),$$

corresponding to the assumed value $p=p_\mu$ we have started with in (24) for obtaining $X_i(\mu), \theta_i(\mu)$. The value $p(\mu)$ obtained from the system (21) and (18-A) by the procedure above will not coincide with the value p_μ in general. Different values p_μ are tried until a consistent result is obtained.

The method, while not entirely analytical, does give a solution, to any accuracy desired, in terms of analytic functions. The accuracy is increased by increasing the number of terms in the series (4).