An Analytic Solution of Heat Conduction Problem With Phase Change

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A solid is bounded by two infinite parallel plates and the whole system is at a temperature T_A which is below the freezing point T_f of the substance. At a certain instant, one of the plates, B, is raised to and maintained at a temperature T_B which is above the freezing point T_f, while the other plate A is maintained at the original temperature T_A such that $T_A < T_f < T_B$. The problem of finding the temperature distribution in the solid and the liquid phases and the position of the interface between the two phases as functions of time is solved by a method in which a solution of the heat conduction problem is constructed by the superposition of solutions such that all the initial, asymptotic and boundary conditions of the problem are satisfied.

1. STATEMENT OF THE PROBLEM

A problem in heat conduction that involves a moving boundary, known as Stefan's problem, is typified by the following one. Consider the water in a wide and very deep lake which is being frozen from above by the surface being in contact with cold air. Let us assume that the temperature of the water at great depths is constant and above the freezing point. As time goes on, the interface between the ice and the water moves downward. The problem is to determine the position and the rate of motion of the interface from the temperatures of the (free) surface at the top and of the water at great depths, the melting point and the heat of fusion of ice. In the case of infinite depth, the above problem has been solved by Neumann[1], and the solution is expressed in terms of error functions which are solutions of the equation of heat conduction and which are appropriate for the boundary condition at infinite distance and the asymptotic condition for long times.

In recent years, many variations of the above problem with a moving boundary have been studied by many authors[2], but it seems that almost all the studies

1. E. R. G. Eckart and R. M. Jr. Drake: *Heat and Mass Transfer* (McGraw Hill. New York, 1959).
2. (1) H. G. Landau, Quarterly Appl. Math.. 8, 81, (1951).
   (2) C. C. Chao and J. H. Meiner, Quarterly Appl. Math.. 14, 214, (1956).
   (3) A. L. Ruoff, Quarterly Appl. Math., 16, 197, (1958).
   (4) E. L. Knuth, Physics of Fluids, 2, 84, (1959).
   (5) G. J. Horvay, Jour. Heat Transfer, 82, 27, (1960).
   (6) T. D. Hamill and S. G. Bankoff, A. I. Ch. E. J., 9, 741, (1963).
   (7) T. D. Hamill and S. G. Bankoff, A. I. Ch. E. J., 3, 177, (1964).

have been concerned with semi-infinite slabs. In many cases, the problem of freezing has been solved by both exact and approximate methods.

Consider next the more realistic case, and hence of more practical value, in which the thickness of the slab is finite.

For the following calculations, let us restate the problem as follows.

A substance is bounded by two infinite, fixed parallel plates A and B and the whole system is originally at the same temperature T_A which is below the freezing point of the substance T_f. At a certain instant, the temperature of the plate B is raised to, and henceforth maintained at, the value T_B which is above the freezing point T_f, while the plate A is maintained at the original temperature T_A. Thus $T_A < T_f < T_B$. The solid near the plate B begins to melt, and the interface surface between the solid and the liquid phases of the substance moves from B toward A and reaches a definite position between A and B when steady state is reached. The rate of the motion of the interface and its position at any time are determined by the temperatures T_A, T_B, T_f, the heat conductivities of the substance in the solid and the liquid phases, the heat of fusion of the substance and the separation distance between the two plates. The problem is to find the temperature distribution in the substance at any time. This then yields the rate of motion and the position of the interface at any time.

While the solution of this problem is completely determined by the basic equation of heat conduction and involves no new principles, nevertheless, the problem seems to have presented some difficulties such that no analytic solution of this problem seems to have been given in the literature. Longwell[3] has recently given a graphical method for treating this problem. We shall refer the reader to this work of Longwell for references to the earlier literature on the subject as well as for the graphical method itself. The purpose of the present work is to describe an attempt to solve the problem by an analytic method. The method described below applies also to the case of the freezing of a fluid between two plates if $T_A > T_f > T_B$.

For simplicity, we shall assume that the thermal conductivity of the solid is a constant k_1 independent of temperature, and similarly that of the liquid is a constant k_2. We shall neglect the change in volume of the substance on melting (or, we allow for the change in volume by a lateral expansion or contraction of the liquid parallel to the plates and assume that this change is small). If the change in volume in melting is taken into account and the plate A is allowed to move to accommodate this change, the problem becomes much more difficult and the solution in the forms (4) and (5) below is no longer valid. This problem is

3. P A. Longwell, AIChE Journal. 4, 53 (1958).

being studied. We shall also neglect any heat transport through convection. The problem becomes then a one-dimensional one, and the equation of heat conduction is

$$\frac{\partial T}{\partial t}=a\frac{\partial^2 T}{\partial x^2}, \qquad a=k/\rho C_v \tag{1}$$

where $T=T(x,t)$ denotes the temperature, t the time, and x the coordinate of a point along a line perpendicular to the plates A and B, a is the thermal diffusivity, k the thermal conductivity, ρ the mass density and C_v the specific heat at constant volume.

2. SOLUTION OF THE PROBLEM

For our present problem, let L be the distance between the two plates, x_1 the distance of a point in the solid phase from the plate A (the outer boundary of the solid phase), x_2 the distance of a point in the liquid phase from the plate A, and $T_1(x_1,t), T_2(x_2,t)$ the temperature at the points x_1, x_2 at time t.

It is convenient to employ dimensionless variables.

Let

$$\begin{aligned}
&\tau_j=(T_j-T_A)/(T_B-T_A), && j=1,\ 2\\
&\tau_f=(T_f-T_A)/(T_B-T_A), &&\\
&\xi_j=x_j/L, && j=1,\ 2\\
&\theta=a_1 t/L^2, &&\\
&\beta=a_2/a_1, &&\\
&\gamma=k_2/k_1, &&
\end{aligned}$$

The ranges of τ_j are ξ_j are as follows:

$$0\leq\tau_1\leq\tau_f, \quad \tau_f\leq\tau_2\leq1, \quad 0\leq\xi_1\leq X, \quad X\leq\xi_2\leq1, \tag{2}$$

where X is the position of the interface. θ ranges from 0 to ∞, and for a given substance, τ_f lies between 0 and 1.

The conduction equation in the phase 1 and phase 2 is respectively

$$\frac{\partial \tau_1}{\partial \theta}=\frac{\partial^2 \tau_1}{\partial \xi_1^2}, \qquad \frac{\partial \tau_2}{\partial \theta}=\beta\frac{\partial^2 \tau_2}{\partial \xi_2^2} \tag{3}$$

We shall construct a solution of a linear differential equation by superposing solutions. Thus we shall choose for $\tau_1(\xi_1,\theta), \tau_2(\xi_2,\theta)$ the forms

$$\tau_1(\xi_1,\theta)=a_1\xi_1+b_1 + \sum_{n=1}^{\infty} d_n \exp(-n^2\pi^2\theta)\sin(n\pi\xi_1)+c_1\Phi\left(\frac{\xi_1}{2\theta^{1/2}}\right), \tag{4}$$

$$\tau_2(\xi_2, \theta) = a_2\xi_2 + b_2 + \sum_{n=1}^{\infty} f_n \exp(-n^2\pi^2\beta\theta) \sin(n\pi\xi_2) + C_2\Phi\left(\frac{1-\xi_2}{2(\beta\theta)^{1/2}}\right), \tag{5}$$

where

$$\Phi(z) = \frac{2}{\pi^{1/2}} \int_0^z \exp(-y^2)\, dy, \tag{6}$$

and $a_1, b_1, c_1, d_n, a_2, b_2, c_2, f_n$ are constants. The arguments of the sine functions in (4) and (5) have been so chosen as to be compatible with the boundary conditions (7),(8), (12) and (17) below. The constants $a_1, b_1, \ldots, f_n$ are to be determined such that the functions τ_1, τ_2 in (4) and (5) satisfy all the boundary, initial and asymptotic conditions of the problem.

These conditions are

At $\theta=0$, $\quad \tau_1(\xi_1, 0)=0$ for $0 \le \xi_1 < 1$, (7)

$$\tau_2(\xi_2, 0) = 1 \text{ at } \xi_2 = 1. \tag{8}$$

As $\theta \to \infty$, the system approaches the steady state. Let X_∞ be the distance, in units of L as in (2), of the interface from the plate A so that

$$\tau_1(\xi_1, \infty) = 0 \text{ at } \xi_1 = 0, \tag{9}$$

$$= \tau_f \text{ at } \xi_1 = X_\infty, \tag{10}$$

$$\tau_2(\xi_2, \infty) = \tau_f \text{ at } \xi_2 = X_\infty, \tag{11}$$

$$= 1 \text{ at } \xi_2 = 1, \tag{12}$$

and the continuity of heat flow across the interface leads to

$$\left.\frac{\partial \tau_1}{\partial \xi_1}\right|_{\xi_1 = X_\infty} - \gamma \left.\frac{\partial \tau_2}{\partial \xi_2}\right|_{\xi_2 = X_\infty} = 0 \tag{13}$$

At finite times θ, we have the boundary conditions

$$\tau_1(\xi_1, \theta) = 0 \text{ at } \xi_1 = 0, \tag{14}$$

$$= \tau_f \text{ at } \xi_2 = X. \tag{15}$$

$$\tau_2(\xi_2, \theta) = \tau_f \text{ at } \xi_2 = X, \tag{16}$$

$$= 1 \text{ at } \xi_2 = 1, \tag{17}$$

where X is the distance, in units of L, of the interface from the plate A at time θ.

The rate at which the interface moves is given by $dX/d\theta$ which must have the same value when calculated from either the liquid or the solid phase, i.e.,

$$\left.\frac{dX}{d\theta}\right|_{\text{from } \tau_1} = \left.\frac{dX}{d\theta}\right|_{\text{from } \tau_2} \tag{18}$$

which is ensured by the conditions (15) and (16). If the latent heat of fusion is Q and the density of the solid phase is ρ_1, then the rate at which heat is absorbed for the melting of the solid of unit cross section is given by the equation of heat balance

$$k_1 \left.\frac{\partial T_1}{\partial x_1}\right|_{x_1=XL} - k_2 \left.\frac{\partial T_2}{\partial x_2}\right|_{x_2=XL} = \rho_1 QL \frac{dX}{dt}, \tag{19}$$

or, in dimensionless units,

$$\left.\frac{\partial \tau_1}{\partial \xi_1}\right|_{\xi_1=X} - r\left.\frac{\partial \tau_2}{\partial \xi_2}\right|_{\xi_2=X} = q\frac{dX}{d\theta}, \tag{19a}$$

where

$$q = Q/C_{v1}(T_B - T_A).$$

The conditions in Equations (7) through (19) are to be satisfied by $\tau_1(\xi_1, \theta)$ and $\tau_2(\xi_2, \theta)$ in (4) and (5).

From (7), we have, for $0 \leq \xi_1 < 1$,

$$0 = a_1\xi_1 + b_1 + \sum_{n=1}^{\infty} d_n \sin(n\pi\xi_1) + c_1 \tag{20}$$

From (9) and (14), we get

$$0 = b_1 \tag{21}$$

From (20), on multiplying by $\sin(m\pi\xi)d\xi$ and integrating from $\xi=0$ to $\xi=1$, one gets

$$d_n = \frac{2}{(n\pi)}\left\{ (-1)^n a_1 - [1-(-1)^n]\, c_1 \right\}, \quad n \neq 0, \tag{22}$$

which determines all the coefficient d_n in (4) in terms of a_1 and c_1.

A word may now be said concerning the choice of the error function and the series in (4). The error function is used to give a better account of the rapid rise of temperature τ_1 near the plate (at τ_B) for small time θ after $\theta=0$. The series then becomes necessary, for without it, (20) and (21) would have led to $a_1=0, c_1=0$.

From (13), (10), and (11) one obtains, respectively,

$$a_1 = r a_2, \tag{23}$$

$$\tau_f = a_1 X_\infty, \tag{24}$$

$$\tau_f = a_2 X_\infty + b_2. \tag{25}$$

From (8), or (12), or (17), we find the same relation

$$1 = a_2 + b_2 \tag{26}$$

From (25) and (26), we get

$$a_2(1-X_\infty)=1-\tau_f. \tag{27}$$

On collecting (21), (23), (24), (25), and (27), we obtain

$$b_1=0, \tag{28}$$

$$a_1=\tau_f+\gamma(1-\tau_f), \tag{29}$$

$$a_2=a_1/\gamma=1-\tau_f+\tau_f/\gamma, \tag{23a}$$

$$X_\infty=\tau_f/a_1, \tag{30}$$

$$b_2=\tau_f(1-1/\gamma). \tag{31}$$

The temperature distribution in the steady state is given, in each phase, by a linear function in the coordinate ξ, namely,

$$\tau_1=a_1\xi_1 \quad \text{for} \quad 0\leq\xi_1\leq X_\infty,$$

$$\tau_2=a_2\xi_2+b_2 \quad \text{for} \quad X_\infty\leq\xi_2<1.$$

Equations (29) and (30) give the position of the interface in the steady state in terms of the temperatures τ_f of the interface and the conductivity ratio $\gamma=k_2/k_1$. The qualitative feature of this result about X_∞ can be seen in the special case in which k_1 and k_2 are the same. In this case, we obtain

$$X_\infty=\tau_f$$

$$X_\infty=L(T_f-T_A)/(T_B-T_A),$$

that is, the thicknesses of the solid and the liquid phases are in the ratio of the temperature differences T_f-T_A and T_B-T_f. In this case the temperature gradients on both sides of the interface in the steady state are the same. In the general case $k_1\neq k_2$, the gradients on the two sides of the interface are inversely proportional to the conductivities as shown by (13).

We shall next proceed to determine the coefficients c_1, c_2 and the f_n in (4) and (5). The boundary conditions at the interface X are: (i), equations (15) and (16) which ensure the equality of τ_1 and τ_2 to the melting point τ_f at X, and (ii), the condition (19a) on the gradients of τ_1 and τ_3 at the interface. These two conditions are independent and not redundant.

Let X_i be the position of the interface at time $\theta_i(0\leq\theta_i<\infty)$. Equations (15) and (16) are

$$\tau_f=a_1X_i+c_1\Phi\left(\frac{X_i}{2\theta_i^{1/2}}\right)+\sum_{n=1} d_n \exp(-n^2\pi^2\theta_i) \sin(n\pi X_i) \tag{32}$$

$$\tau_f=a_2X_i+b_2+c_2\Phi\left(\frac{1-X_i}{2(\beta\theta_i)^{1/2}}\right)+\sum_{n=1} f_n \exp(-n^2\pi^2\beta\theta_i) \sin(n\pi X_i) \tag{33}$$

Condition (i) is then

$$a_2X_i + b_2 + c_2\Phi\left(\frac{1-X_i}{2(\beta\theta_i)^{1/2}}\right) + \sum_{n=1} f_n \exp(-n^2\pi^2\beta\theta_i)\sin(n\pi X_i)$$

$$= a_1X_i + c_1\Phi\left(\frac{X_i}{2\theta_i^{1/2}}\right) + \sum_{n=1} d_n \exp(-n^2\pi^2\theta_i)\sin(n\pi X_i) \quad (34)$$

Condition (ii) can be expressed as follows. If (32), (33) are written in the form

$$F(X_i,\theta_i)=0,\quad G(X_i,\theta_i)=0, \quad (35)$$

then the rate of motion of the interface X_j is given by

$$\frac{dX_i}{d\theta_i} = -\frac{(\partial F/\partial\theta_i)\,x_i}{(\partial F/\partial X_i)_{\theta i}} = \frac{(\partial G/\partial\theta_i)\quad x_i}{(\partial G/\partial X_i)_{\theta i}} \quad (36)$$

If we denote

$$\psi_1 = \frac{1}{(\pi\theta_i)^{1\ 2}}\exp\left(-\left[\frac{X_i^2}{4\theta_i}\right]\right),$$

$$\psi_2 = \frac{1}{(\pi\beta\theta_i)^{1\ 2}}\exp\left(-\left[\frac{(1-X_i)^2}{4\beta\theta_i}\right]\right), \quad (37)$$

$$B(\mathrm{i}) = \sum_{n=1} (n\pi)^2\ d_n \exp(-n^2\pi^2\theta_i)\sin(n\pi X_i),$$

$$G(\mathrm{i}) = \sum_{n=1} (n\pi)\, d_n \exp(-n^2\pi^2\theta_i)\cos(n\pi X_i),$$

$$D_n(\mathrm{i}) = (n\pi)^2 \exp(-n^2\pi^2\beta\theta_i)\sin(n\pi X_i),$$

$$E_n(\mathrm{i}) = n\pi \exp(-n^2\pi^2\beta\theta_i)\cos(n\pi X_i) \quad (38)$$

$$M(i) = \frac{-c_1\frac{\mathrm{Xi}}{2\theta_i}\psi_1 - B(i)}{a_1 + c_1\psi_1 + C(i)} \quad (39)$$

$$\mathrm{N(i)} = \frac{-c_2\frac{1-X_i}{2\theta_i}\psi_2 - \beta\sum_n D_n(i)f_n}{a_2 - c_2\psi_2 + \sum_n E_n(i)f_n} \quad (39)$$

then (36) becomes

$$-\frac{dX_i}{d\theta_i} = M(i) = N(i) \quad (40)$$

and the equation (19a) becomes, on using the relation (23),

$$c_1\psi_1 + C(i) - r\{-c_2\psi_2 + \sum_{n=1} E_n(i)f_n\} = -qM(i) \quad (41)$$

This equation gives the condition for the slopes of the τ_1 and τ_2 curves at the interface Xi at time θ_i. Equations (41) and (34) are linearly independent.

In all the preceding relations (32) —(41), the coefficients $c_1, c_2, f_1, f_2, \ldots f_n, \ldots$ are yet undetermined. To determine them, the following procedure is suggested. Equation (32), which now reads, on account of (22),

$$\tau_f = a_1 X_i + c_1 \Phi\left(\frac{X_i}{2(\theta_i)^{1/2}}\right) + \sum_{n=1} \frac{2}{(n\pi)}\left\{(-1)^n a_1 - [1-(-1)^n]\, c_1\right\} \times \exp(-n^2\pi^2\theta_i)\ \sin(n\pi X_i), \tag{32a}$$

can be regarded as a functional relation between X_i and θ_i for any given value of c_1, i.e., for any assumed value of c_1, say, $c_1(1)$, one can obtain a X_i (1) vs. θ_i (1) curve. Let us take m pairs of values $\{X_i(1), \theta_i(1)\}. i=1, 2, \ldots m$. On inserting these m pairs of values in turn into (33), one obtains m equations. Let us truncate the series in f_n at $n=m-1$. From this system of m equations, one obtains c_2, f_1, $f_2, \ldots f_{m-1}$. Let this set of coefficients (corresponding to $c_1=c_1$ (1) assumed in (32a)) be denoted by

$$c_1(1),\ c_2(1),\ f_1(1), \ldots f_{m-1}(1). \tag{42}$$

If this set were the correct solution to the original problem, these values (42) and the m pairs of values $\{x_i(1), \theta_i(1)\}$ would also satisfy (41). The method of finding the solution is then to try other values $c_1=c_1(s)$, leading by (32a) to the values $\{x_i(s), \theta_i(s)\}$ and by (33) to

$$c_1(s), c_2(s), f_1(s), \ldots f_{m-1}(s), \tag{43}$$

until these values in (43) satisfy equation (41).

An alternative to the above procedure is as follows. Start with an assumed value c_1 (1) for c_1 in (32a). Take m pairs of values $\{X(1), \theta_i(1)\}, i=1, 2, \ldots m$, corresponding to this value $c_1=c_1$ (1). Substitute these m pairs in turn into equations (33) and (41) (in the calculation of M(i) in (41), the assumed value $c_1=c_1(1)$ is used, while the c_1 on the left hand side of (41) is regarded as an unknown. (41) is then linear in c_1,) thereby yielding $2m$ equations. On truncating the series in f_n at $n=2m-2$, these $2m$ equations can be regarded as a linear system for the $2m$ coefficients

$$c_1, c_2, f_1, f_2, \ldots f_{2m-2}. \tag{44}$$

The value obtained for c_1 from these systems in general will not coincide with the value $c_1=c_1$ (1) assumed in (32a). To obtain the solution of the problem, other sets of $\{X_i(s), \theta_i(s)\}$ corresponding to $c_1=c_1$ (s) in (32a) are tried until the resulting c_1 in (44) obtained from (33) and (41) agrees with the value c_1 assumed.

Both procedures, essentially the same, described above can only be justified from the point of view of "fitting point-by-point" or of "self-consistency."

In principle, either procedure can be expected to give the desired solution only in the limit when the number m of points $\{X_i, \theta_i\}$ becomes infinite. It certainly is not good for a small value of m, such as $m=5$. In practice, however, one may obtain a good approximation to the solution by judiciously choosing a finite number of pairs $\{X_i, \theta_i\}$ for well-spaced X_i between $X_i=0$ and $X_i=X_\infty$.

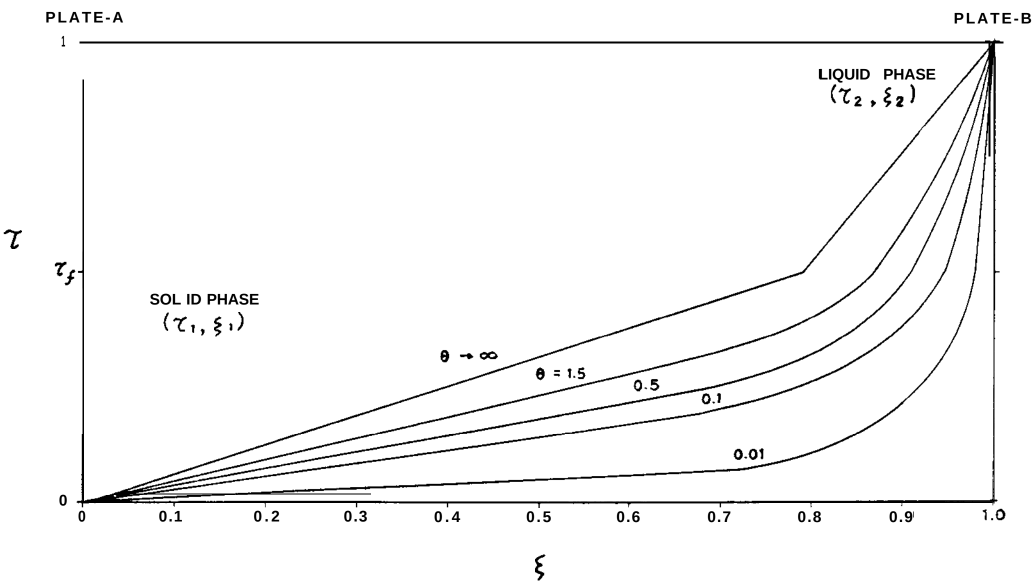

Fig. 1.

We may now say a few words about the choice of the functions in the representation of $\tau_2(\xi, \theta)$ in (5). The use of the error function $c_2\Phi$ alone would have met the conditions (8), (10), (11), (12), (17). But from (34), it is seen that, without the generality provided by the series in f_n, it is not possible to satisfy the condition (34), nor the condition (41). On the other hand, with the series in f_n, it is not absolutely necessary to include the error function $c_2\Phi$ in (5). It has been included in (5) since it alone would give a good approximation to the τ_2 behavior for $\xi_2 < 1$ and small θ, thus leaving for the series in f_n the main burden of satisfying the boundary conditions at the interface.

That the method for solving the original melting problem is so roundabout **is** understandable on the following consideration. For convenience let us refer to the accompanying figure. For any given time θ_i, the temperature distributions $\tau_1(\xi), \tau_2(\xi)$, being given by a second order partial differential equation, have to be solved subject to the boundary conditions at $\xi_1 = 0$, $\xi_2 = 1$, and the conditions (34) and (19a) at the interface X_i which is moving and which has yet to be found for a given time θ_i. This moving boundary is the cause of difficulty of obtaining, in closed analytic form, the solution of the problem which consists of a continuous family of curves as indicated in the accompanying figure.