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On the Time-Reversal Invariance of the Fundamental Commutation Relation

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We check the time-reversal invariance of the fundamental commutation relation $[p, x] = -i\hbar$. Although it has been checked in the active picture, where the transformation affects the states but not the operators, it is a puzzling issue in the passive picture, where the transformation affects the operators but not the states. It is puzzling because $[p', x'] = [-p, x] = i\hbar$ in the passive picture. At first sight, it seems that the fundamental commutation relation is not time-reversal invariant. In fact, in both older and more recent textbooks on quantum mechanics, the time-reversal invariance of the fundamental commutation relation has never been explicitly clarified in the passive picture. We point out that the claims in standard textbooks concerning the time-reversal invariance of the fundamental commutation relation in the passive picture are misleading. We show that $[p, x]' = [p', x'] = [-p, x] = i\hbar$, if the time-reversal operator is unitary. Hence the fundamental commutation relation is not time-reversal invariant if the time-reversal operator is unitary. On the other hand, $[p, x]' = [x', p'] = [x, -p] = -i\hbar$, if the time-reversal operator is antiunitary. We conclude that the fundamental commutation relation is time-reversal invariant, provided that the time-reversal operator is antiunitary. The important fact $[p, x]' = [x', p']$, which is the origin of the difficulties, is not appreciated in standard textbooks on quantum mechanics. The present discussion provides a more fundamental justification for taking the time-reversal operator to be antiunitary, which applies to particles with arbitrary spin.

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I. INTRODUCTION

At the core of quantum mechanics lies the fundamental commutation relation

$$[p, x] = -i\hbar\,. \tag{1}$$

Relation (1) follows from the Einstein–de Broglie relations $E = h\nu$, $p = \frac{h}{\lambda}$, which are an expression of the wave-particle duality. Many puzzling features of quantum behavior result from the fundamental commutation relation (1). It is the fundamental commutation relation (1) that stands behind the Heisenberg uncertainty relation $\Delta p \Delta x \geq \frac{h}{4\pi}$. As pointed out by Dirac [1], relation (1) gives the solution to the problem of finding the quantum conditions for all those dynamical systems which have a classical analogue. For this reason, Dirac called the relation (1) "the fundamental quantum condition".

As a cornerstone of quantum mechanics, the fundamental commutation relation must be invariant under the Lorentz transformations. The aim of this paper is to check the time-reversal invariance of the fundamental commutation relation. Under the time-reversal transformation, the space-time coordinates transform as $t' = -t$, $\boldsymbol{r'} = \boldsymbol{r}$. In classical physics, the momentum transforms as $p' = -p$. The sign change for the momentum is obvious from its relation to the velocity $v = \frac{dx}{dt}$. At first sight, it seems that the fundamental commutation relation (1) is not invariant under the time-reversal transformation, because

$$[p', x'] = i\hbar \tag{2}$$

in the time-reversed coordinate system. This led Gottfried to make the following statement [2]: "the transformation $x \to x, p \to -p$ does not leave the canonical commutation rules invariant." In the viewpoint of Sakurai, the time-reversal invariance of the fundamental commutation relation can be restored by treating the number i as an operator whose time-reversed operator is $-i$. Sakurai [3] asserted that "In quantum mechanics, we have $[x_i, p_j] = i\delta_{ij}$ which is preserved since $x_i \xrightarrow{T} x_i$, $p_j \xrightarrow{T} -p_j$ and $i \xrightarrow{T} -i$." Messiah was also led to the same conclusion. In his book [4], Messiah wrote down several rules concerning antiunitary transformations: "any relation between vectors and/or operators is also satisfied by the transforms of these quantities if we replace all coefficients by their complex conjugates. In other words, the transformation K conserves equations between vectors and/or operators if we agree to treat the coefficients involved as operators. For example the commutation relation $[q, p] = ih$, $[J_x, J_y] = i\hbar J_z$ transform respectively into $[\hat{q}, \hat{p}] = -i\hbar$, $[\hat{J}_x, \hat{J}_y] = i\hbar \hat{J}_z$." "any algebraic relation between the observables of the system is conserved in the transformation if T is linear, and is replaced by the complex conjugate relation if T is antilinear." "Any observable B is a certain real function $F(\xi)$ of the fundamental observables $\xi_1, \xi_2, \ldots, \xi_n, \ldots$, of the system. Its transform is necessarily $B' \equiv F(\xi')$." "either the transformation conserves the fundamental commutation relation, or it changes the sign. In the first case, the operator T associated with the transformation is linear; in the second case, it is antilinear." "T is unitary if the transformation conserves the commutation relations antiunitary if it changes their sign." Messiah was thus led to make the following statement concerning the time-reversal transformation: "The transformation changes the sign of the commutation relation, and K is therefore an antiunitary operator." It is our opinion that these claims are misleading. The present paper aims to clarify a few issues and to point out the origin of the difficulty.

II. TIME-REVERSAL INVARIANCE OF THE FUNDAMENTAL COMMUTATION RELATION

To begin with, we clarify what is meant by the time-reversed operator. Mathematical entities such as states and operators are not directly accessible to physical observation. The comparison with measurement is made in terms of the expectation value. The expectation value of an linear operator A is given by

$$\begin{aligned} \langle A \rangle &= \langle \Psi(t), A\Psi(t) \rangle \\ &= \int d^3\boldsymbol{r} \Psi^\dagger(\boldsymbol{r}, t) A \Psi(\boldsymbol{r}, t) \,. \end{aligned} \tag{3}$$

Under the time-reversal transformation the expectation value $\langle A\rangle$ is transformed into $\langle A\rangle'$. One can adopt two equivalent pictures to interpret the transformation. In the first picture, the transformation affects the states but not the operators. Each state is transformed into a time-reversed state

$$\Psi'(t') = T\Psi(t)\,, \tag{4}$$

where T is the time-reversal operator. Thus the expectation value $\langle A\rangle$ is transformed into

$$\langle A\rangle' = \langle T\Psi(t), AT\Psi(t)\rangle\,. \tag{5}$$

This picture is called the active transformation picture [5]. In the second picture, the transformation affects the operators but not the states. Each operator A is transformed into a time-reversed operator A'. Thus the expectation value $\langle A\rangle$ is transformed into

$$\langle A\rangle' = \langle \Psi(t), A'\Psi(t)\rangle\,. \tag{6}$$

This picture is called the passive transformation picture [5]. It is well-known that [6] symmetry operators fall into two classes: unitary and antiunitary. In the case that T is unitary, (5) becomes

$$\begin{aligned}\langle A\rangle' &= \langle \Psi(t), T^\dagger AT\Psi(t)\rangle \\ &= \langle \Psi(t), T^{-1}AT\Psi(t)\rangle\,.\end{aligned} \tag{7}$$

A comparison of (6) and (7) shows that

$$A' = T^{-1}AT\,. \tag{8}$$

On the other hand, if T is antiunitary, then T can be expressed as [7]

$$T = UK\,, \tag{9}$$

where U is a unitary operator and K is the complex-conjugation operator. Note that $K^2 = 1$, and therefore

$$T^{-1} = KU^\dagger\,. \tag{10}$$

In this case, (5) becomes

$$\begin{aligned}\langle A\rangle' &= \langle UK\Psi(t), AUK\Psi(t)\rangle \\ &= \langle A^\dagger UK\Psi(t), UK\Psi(t)\rangle \\ &= \langle U^\dagger A^\dagger UK\Psi(t), K\Psi(t)\rangle \\ &= \langle KU^\dagger A^\dagger UK\Psi(t), \Psi(t)\rangle^* \\ &= \langle \Psi(t), KU^\dagger A^\dagger UK\Psi(t)\rangle \\ &= \langle \Psi(t), T^{-1}A^\dagger T\Psi(t)\rangle\,.\end{aligned} \tag{11}$$

A comparison of (6) and (11) yields

$$A' = T^{-1}A^\dagger T\,. \tag{12}$$

Note that we arrive at (12) without using the hermitian conjugate operator of T. In fact, the hermitian conjugate operator of an antiunitary operator does not exist.

It is instructive to investigate the time-reversal transformation of the product of two operators. The expectation value of the product AB is

$$\langle AB\rangle = \langle \Psi(t), AB\Psi(t)\rangle\,. \tag{13}$$

In the active transformation picture, $\langle AB\rangle$ is transformed into

$$\langle AB\rangle' = \langle T\Psi(t), ABT\Psi(t)\rangle\,, \tag{14}$$

whereas in the passive transformation picture

$$\langle AB\rangle' = \langle \Psi(t), (AB)'\Psi(t)\rangle\,. \tag{15}$$

In the case that T is unitary, (14) becomes

$$\begin{aligned}\langle AB\rangle' &= \langle \Psi(t), T^{-1}ABT\Psi(t)\rangle \\ &= \langle \Psi(t), T^{-1}ATT^{-1}BT\Psi(t)\rangle\,.\end{aligned} \tag{16}$$

A comparison of (15) and (16) yields

$$\begin{aligned}(AB)' &= T^{-1}ATT^{-1}BT \\ &= A'B'\,.\end{aligned} \tag{17}$$

Equation (8) has been used in arriving at (17). In the case that T is antiunitary, (14) becomes

$$\begin{aligned}\langle AB\rangle' &= \langle UK\Psi(t), ABUK\Psi(t)\rangle \\ &= \langle U^\dagger B^\dagger A^\dagger UK\Psi(t), K\Psi(t)\rangle \\ &= \langle K\Psi(t), U^\dagger B^\dagger A^\dagger UK\Psi(t)\rangle^* \\ &= \langle \Psi(t), KU^\dagger B^\dagger A^\dagger UK\Psi(t)\rangle \\ &= \langle \Psi(t), T^{-1}B^\dagger TT^{-1}A^\dagger T\Psi(t)\rangle\,.\end{aligned} \tag{18}$$

A comparison of (15) and (18) shows that

$$\begin{aligned}(AB)' &= T^{-1}B^\dagger TT^{-1}A^\dagger T \\ &= B'A'\,,\end{aligned} \tag{19}$$

where (12) has been used. Now comes the crucial point: *under an antiunitary transformation, the transform of the product of two operators equals the product of the transform of the factors in the reverse order* . This important fact has not been pointed out in standard textbooks on quantum mechanics. Thus Messiah's claim that the transform of $B \equiv F(\xi)$ is necessarily $B' \equiv F(\xi')$ is wrong. With the aids of (17) and (19), we see that

$$[A, B]' = [A', B'] , \tag{20}$$

if T is unitary, or

$$[A, B]' = [B', A'] , \tag{21}$$

if T is antiunitary.

We are now in a position to check the time-reversal invariance of the fundamental commutation relation (1). In the active picture, the time-reversal invariance of the fundamental commutation relation can be easily seen. Sakurai [8] showed that

$$[p, x]T\Psi(t) = -i\hbar T\Psi(t) \tag{22}$$

by virtue of the fact that T is antiunitary. However in the passive picture, as we have mentioned in section I, it is a puzzling issue. Gottfried claimed that the fundamental commutation relation is not invariant under the time-reversal transformation, whereas Sakurai and Messiah asserted that the time-reversal invariance of the fundamental commutation relation can be restored by treating the number i as an operator whose time-reversed operator is $-i$. In the following, we will see that such claims are misleading. In fact, in both older and more recent textbooks on quantum mechanics, the time-reversal invariance of the fundamental commutation relation has never been clarified explicitly in the passive picture. In quantum mechanics, the expectation values play the role of classical variables. So in the passive picture, it is reasonable to expect that

$$\langle \Psi(t), p'\Psi(t)\rangle = -\langle \Psi(t), p\Psi(t)\rangle \tag{23}$$

and

$$\langle \Psi(t), x'\Psi(t)\rangle = \langle \Psi(t), x\Psi(t)\rangle . \tag{24}$$

Thus we have $p' = -p$ and $x' = x$ in the passive picture. In the case that T is unitary, we have

$$\begin{aligned} [p, x]' &= [p', x'] \\ &= [-p, x] \\ &= i\hbar, \end{aligned} \tag{25}$$

where (1) and (20) have been used. Thus the fundamental commutation relation is not time-reversal invariant if T is unitary. In the case that T is antiunitary, we have

$$\begin{aligned} [p, x]' &= [x', p'] \\ &= [x, -p] \\ &= [p, x] \\ &= -i\hbar. \end{aligned} \tag{26}$$

Equation (26) shows that *the fundamental commutation relation is time-reversal invariant, provided that T is antiunitary.* Note that in the passive picture, the fundamental commutation relation is preserved without treating the number i as an operator whose time-reversed operator is $-i$. Thus we conclude that the claims of Sakurai and Messiah are wrong. The important fact $[p, x]' = [x', p']$ has not been appreciated in standard textbooks on quantum mechanics. The identity (19) is the origin of the difficulty. In a standard textbook on quantum mechanics [7], it has been shown that the time-reversal operator for particles obeying the Schrödinger equation is antiunitary. The present demonstration provides a more fundamental justification for taking the time-reversal operator to be antiunitary, which applies to particles with arbitrary spin.

III. CONCLUSIONS

In this paper, we check the time-reversal invariance of the fundamental commutation relation. We clarify a few issues and point out the origin of the difficulty. We show that the antiunitary transformation of the product of two operators is given by the product of their antiunitary transformation taken in reverse order. As a matter of fact, this is the key to the controversy. We point out that the fundamental commutation relation is invariant under the time-reversal transformation, provided that the time-reversal operator is antiunitary.

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