# The Decay Properties of a Single-photon in Linear Media

J. R. Choi*

*Department of Electronic Physics, College of Natural Science,*
*Hankuk University of Foreign Studies, Yongin 449-791, Korea*
(Received July 6, 2002)

We quantized light propagating in linear media in the Coulomb gauge. The conductivity of the media leads to a dissipation of energy for the photons. The quantum mechanical energy of a single-photon decreases continuously and exponentially over time. The zero point energy also decreases in a similar way. The rate of the energy loss coincides with the classical analysis.

PACS numbers: 03.65.Ge, 03.65.-w, 42.50.Ct

## I. INTRODUCTION

The dispute between the wave and particle theories of light from the early days of physics history was ended by the finding of the quantum theory that every quantum mechanical phenomenon has the wave particle duality property. Although outstanding advances have been accomplished in optics up till now, yet probably many properties of light still remain to be uncovered. The main purpose of this paper is to investigate whether the energy of a single-photon can vary continuously or not.

To manage light precisely, it needs to be quantized. The method of quantizing light propagating in free space or in a material like a transparent insulator is well known and appears in most quantum optics text books. The solution is just the same as the case of an ordinary simple harmonic oscillator. The quantization of damped photons is somewhat more subtle and can be found in Refs. [1] and [2]. It involves the quantum theory description of a system interacting with a reservoir having a large number of degrees of freedom. However, the Hamiltonian they used in the development of the theory is an assumed one, rather than having been derived in a clear way. We will quantize light propagating in a homogeneous linear media, where the conductivity cannot be neglected, using a Hamiltonian that is explicitly derived from the fundamental Maxwell's equations. We choose the Coulomb gauge for simplicity. The electric and magnetic fields can then be represented simply by expanding the vector potential, since the scalar potential is zero in media that have no charge density. This gauge is convenient for treating purely transverse waves as in this case. Maxwell's equations give a damped wave equation represented by the vector potential. The light can be quantized with the Hamiltonian for the damped wave equation, by the invariant operator method [3–8] first introduced by Lewis [9]. The invariant operator can be easily derived from the fact that its differentiation with respect to time vanishes. We can obtain a wave function describing damped photons using the eigenstate of the invariant operator. The wave function is a state corresponding to a finite number of photons in an electromagnetic field. We will find the quantum mechanical properties of the electromagnetic field and the mechanical energy expectation value of a single-photon. In the last section we will explain the properties of a damped single-photon using our ideas.

## II. A CLASSICAL SOLUTION FOR THE VECTOR POTENTIAL

By virtue of the invariance under gauge transformations, we are free to choose the kind of gauge for treating electromagnetic phenomena. Although none of the chosen gauges will affect the quantities of the electric and magnetic fields, the description of a particle moving in an electromagnetic field requires the scalar and vector potential. The photon is called a gauge particle since the scalar and vector potential describe the photon quantum mechanically.

The linear media which we treat in this paper not only satisfies $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{H} = \mathbf{B}/\mu$ between the fields but also $\mathbf{J} = \sigma\mathbf{E}$ for the current density, where $\epsilon$, $\mu$, and $\sigma$ are the electric permittivity, magnetic permeability, and conductivity of the media, respectively.

We choose the Coulomb gauge for convenience, $\nabla \cdot \mathbf{A} = 0$. The scalar and vector potentials then satisfy the equations

$$\nabla^2\Phi = -\frac{\rho}{\epsilon}, \tag{1}$$

$$\Box^2\mathbf{A} = -\mu\mathbf{J}_t, \tag{2}$$

where $\rho$ is the charge density, $\mathbf{J}_t$ is the transverse component of the current density and $\Box^2 \equiv \nabla^2 - \epsilon\mu(\partial^2/\partial t^2)$ is called the d'Alembertion. Then, for $\rho = 0$, we obtain $\Phi = 0$ and the damped wave equation for $\mathbf{A}$ is given by

$$\nabla^2\mathbf{A} - \sigma\mu\frac{\partial\mathbf{A}}{\partial t} - \epsilon\mu\frac{\partial^2\mathbf{A}}{\partial t^2} = 0. \tag{3}$$

An example of the same damped wave equation for the above equation in another gauge is found in Ref. [10]. Note that choosing another kind of gauge, for instance the Lorentz gauge, will not always give this damped wave equation for the potential.

Although any gauge gives the same physical result, the treatment of the electromagnetic phenomena must be altered significantly in accordance with the kind of gauge chosen. Here we consider the case where there are no charge distributions, but the conductivity of the media is not neglected. Then we need only treat the damped wave equation for $\mathbf{A}$, Eq. (3), to solve the light quantization problem. Let's separate $\mathbf{A}(\mathbf{r}, t)$ into $\mathbf{r}$ and $t$ components as follows:

$$\mathbf{A}(\mathbf{r}, t) = \sum_l \mathbf{u}_l(\mathbf{r})q_l(t), \tag{4}$$

where the subscript $l$ indicates the various frequency modes of the oscillating light wave. We can insert $\mathbf{A}(\mathbf{r}, t)$ from the above equation into Eq. (3), to obtain the following equations for the respective components of the vector potential [1] :

$$\nabla^2 \mathbf{u}_l(\mathbf{r}) + \frac{\omega_l^2}{c^2}\mathbf{u}_l(\mathbf{r}) = 0, \tag{5}$$

$$\frac{\partial^2 q_l(t)}{\partial t^2} + \frac{\sigma}{\epsilon}\frac{\partial q_l(t)}{\partial t} + \omega_l^2 q_l(t) = 0, \tag{6}$$

where the separation constant $\omega_l$ means the oscillation frequency when the damping term vanishes, and $c \equiv 1/\sqrt{\epsilon\mu}$ is the velocity of the wave in the media.

We now consider light confined in a rectangular cube with side $L$. The wall of the cube is composed of a perfect conductor and the contained media is optically linear. From the condition that at the walls the tangential component of $\mathbf{E}$ and the perpendicular component of $\mathbf{B}$ vanish, the electromagnetic field inside the cube becomes a standing wave. The solution for $\mathbf{u}_l(\mathbf{r})$ can be determined by the normal mode condition. Thus the polarization mode of $\mathbf{u}_l(\mathbf{r})$ for the $x$ direction is given by

$$\mathbf{u}_{l,l'}(\mathbf{r}) = \frac{2}{\sqrt{V}}\hat{\varepsilon}_x \sin\left(\frac{l\pi y}{L}\right)\sin\left(\frac{l'\pi z}{L}\right), \tag{7}$$

where $l,\ l' = 1, 2, 3, \cdots$, $V$ is the volume of the cube and $\hat{\varepsilon}_x$ is the unit vector in the direction of $x$.

Let us confine the focus only to the case where there is under-damping, i.e. for $\sigma^2/(4\epsilon^2) < \omega_l^2$. Then the classical solution of Eq. (6) is

$$q_l(t) = C_l \exp\left(-\frac{\sigma}{2\epsilon}t\right)\cos(\Omega_l t + \phi_l), \tag{8}$$

where $C_l$ and $\phi_l$ are integral constants and $\Omega_l = \sqrt{\omega_l^2 - \sigma^2/(4\epsilon^2)}$.

## III. HAMILTONIAN

We can represent the mechanical energy with the $\mathbf{E}$ and $\mathbf{B}$ field as follows:

$$E = \frac{1}{2}\int_V \left(\epsilon\mathbf{E}^2 + \frac{1}{\mu}\mathbf{B}^2\right) d^3\mathbf{r}. \tag{9}$$

Since $\Phi = 0$ in the above relation, the $\mathbf{E}$ and $\mathbf{B}$ fields are respectively given by,

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}, \tag{10}$$

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{11}$$

After performing integration with the aid of the above two equations as well as Eq. (4), Eq. (9) becomes

$$E = \frac{\epsilon}{2}\sum_l (\dot{q}_l^2 + \omega_l^2 q_l^2) = \sum_l E_l. \tag{12}$$

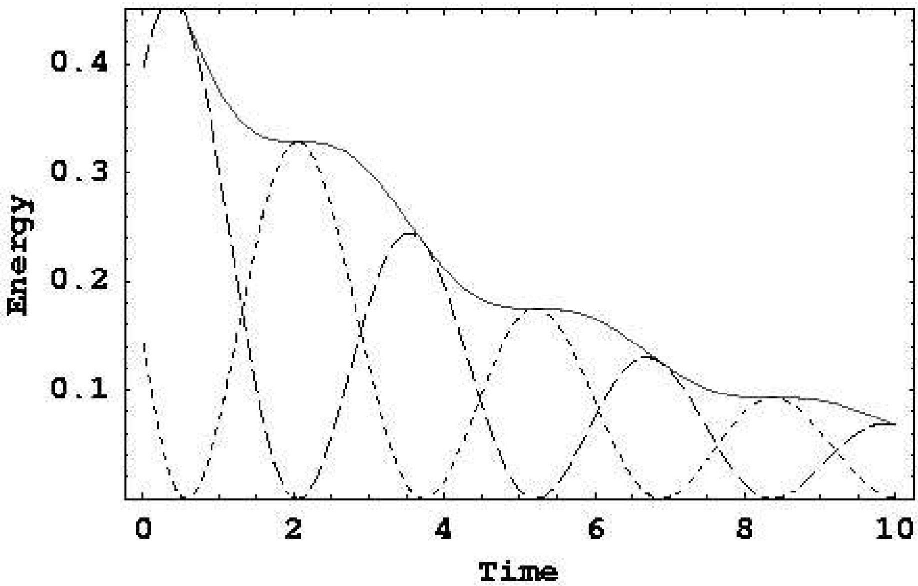


FIG. 1: Mechanical($E_l$, solid line), kinetic($T_l$, long dotted line) and potential($V_l$, short dotted line) energies given in Eqs. (13)-(15). We used $\omega_l = 1$, $\epsilon = 1$, $\sigma = 0.2$, $C_l = 1$ and $\phi_l = 1$.

Note that this equation does not contain $\mathbf{u}_l(\mathbf{r})$. By inserting Eq. (8) and its time derivative into Eq. (12), we can easily confirm that the kinetic and potential energy of mode $l$ can be calculated to be

$$T_l = \frac{1}{2} C_l^2 \epsilon \omega_l^2 \exp\left(-\frac{\sigma}{\epsilon} t\right) \cos^2(\Omega_l t + \phi_l - \delta_l), \tag{13}$$

$$V_l = \frac{1}{2} C_l^2 \epsilon \omega_l^2 \exp\left(-\frac{\sigma}{\epsilon} t\right) \cos^2(\Omega_l t + \phi_l), \tag{14}$$

where $\delta_l = \tan^{-1}(2\Omega_l \epsilon / \sigma)$. Using the above two equations, Eq. (12) can be represented as

$$E = \frac{1}{2} \sum_l C_l^2 \epsilon \omega_l^2 \exp\left(-\frac{\sigma}{\epsilon} t\right) \left\{ 1 + \frac{\sigma}{2\omega_l \epsilon} \cos[2(\Omega_l t + \phi_l) - \delta_l] \right\}. \tag{15}$$

In Fig. 1, we depicted the energies in Eqs. (13)-(15) as a function of time. This figure says that the mechanical energy dissipates as time goes by, especially for a large kinetic energy. This implies that the energy dissipation is strongly related to the velocity of the oscillator. The rate of energy dissipation becomes large as the velocity increases. At the instant when the velocity is zero, i.e. at the turning point of the oscillation, the energy does not dissipate at all.

The mean value of the mechanical energy with respect to time can be obtained by performing a time-average of Eq. (15),

$$\langle E\rangle \simeq \frac{1}{2}\sum_l C_l^2 \epsilon\omega_l^2 \exp\Big(-\frac{\sigma}{\epsilon}t\Big). \tag{16}$$

In this calculation we assume that the damping is small ($\sigma \ll 2\epsilon\omega_l$), so that the variation of $\exp(-\sigma t/\epsilon)$ can be neglected when compared to that of $\cos^2(\Omega_l t + \phi_l - \delta_l)$ [11]. From the above equation we can say that the mean value of the mechanical energy exponentially decreases over time.

When quantizing light propagating in a vacuum we can determine the Hamiltonian from the expression of the energy [1]. However, in case of a dissipating system, i.e. $\sigma \neq 0$, the energy can not give the Hamiltonian directly, since the Hamiltonian and the energy operator are not the same [12]. In a dissipating system, to obtain an expression for the Hamiltonian we must use the equation of motion, i.e. in our case Eq. (6), but we do not need to consider Eq. (5), since $\mathbf{u}_l(\mathbf{r})$ is not contained in the energy expression, as mentioned previously. We can specify the Hamiltonian satisfying Eq. (6) for each frequency mode as

$$H_l(\hat{q}_l,\hat{p}_l,t) = \exp\Big(-\frac{\sigma}{\epsilon}t\Big)\frac{\hat{p}_l^2}{2\epsilon} + \frac{1}{2}\exp\Big(\frac{\sigma}{\epsilon}t\Big)\epsilon\omega_l^2\hat{q}_l^2, \tag{17}$$

where $\hat{p}_l$ is the canonical momentum defined as $-i\hbar(\partial/\partial\hat{q}_l)$. We can check that the above Hamiltonian corresponds to Eq. (6) by directly using Hamilton's equations of motion. We can not find another Hamiltonian that gives the same classical equation of motion in one dimension as that of Eq. (6). The Hamiltonian in Eq. (17) is unique for a one dimensional problem, as far as we know.

However, for a two dimensional problem, there is another Hamiltonian which gives the damped equation of motion known as the Feshbach-Tikochinsky Hamiltonian [13]:

$$H = \frac{1}{\epsilon}\hat{p}_1\hat{p}_2 - \frac{1}{2\epsilon}\sigma(\hat{q}_1\hat{p}_1 - \hat{q}_2\hat{p}_2) + \left(\epsilon\omega^2 - \frac{\sigma^2}{4\epsilon}\right)\hat{q}_1\hat{q}_2. \tag{18}$$

One of the corresponding equations of motion for the Feshbach-Tikochinsky Hamiltonian is same as that in Eq. (6), and the other is the one for an amplified harmonic oscillator:

$$\frac{\partial^2 q_2(t)}{\partial t^2} - \frac{\sigma}{\epsilon}\frac{\partial q_2(t)}{\partial t} + \omega^2 q_2(t) = 0. \tag{19}$$

Since the photon is a boson, the canonical variables satisfy the boson commutation condition, $[\hat{q}_l,\hat{p}_{l'}] = i\hbar\delta_{ll'}$. The total Hamiltonian is a sum of individual terms for each mode, $H = \sum_l H_l$. For most natural light, the electromagnetic field is a collection of numerous independent harmonic oscillators.

## IV. INVARIANT OPERATOR

The invariant operator $\hat{I}_l$ of the system can be obtained from the condition that $d\hat{I}_l/dt = 0$ [4, 14]:

$$\hat{I}_l(\hat{q}_l, \hat{p}_l, t) = \frac{1}{2}\left\{\exp\left(\frac{\sigma}{\epsilon}t\right)\epsilon\Omega_l^2\hat{q}_l^2 + \frac{1}{\epsilon}\exp\left(-\frac{\sigma}{\epsilon}t\right)\left[\hat{p}_l + \frac{\sigma}{2}\exp\left(\frac{\sigma}{\epsilon}t\right)\hat{q}_l\right]^2\right\}. \tag{20}$$

We introduce the following annihilation operator

$$\hat{a}_l(t) = \sqrt{\frac{1}{2\hbar\epsilon\Omega_l}}\left\{\left(\epsilon\Omega_l + i\frac{\sigma}{2}\right)\exp\left(\frac{\sigma}{2\epsilon}t\right)\hat{q}_l + i\exp\left(-\frac{\sigma}{2\epsilon}t\right)\hat{p}_l\right\}, \tag{21}$$

and the creation operator $\hat{a}_l^\dagger(t)$, defined as the conjugate of $\hat{a}_l(t)$. These ladder operators have the usual properties that the annihilation operator $\hat{a}_l$ annihilates a photon and the creation operator $\hat{a}_l^\dagger$ creates a photon. These operators are not Hermitian; they satisfy the commutation relations $[\hat{a}_l(t), \hat{a}_{l'}^\dagger(t)] = \delta_{ll'}$. Note that the ladder operators found in Refs. [1] and [2] do not satisfy these commutation relations. If we denote the eigenstate and eigenvalue of the invariant operator as $|u_{n,l}(t)\rangle$ and $\lambda_{n,l}$, respectively; the eigenvalue equation for $\hat{I}_l$ can be represented as

$$\hat{I}_l|u_{n,l}(t)\rangle = \lambda_{n,l}|u_{n,l}(t)\rangle. \tag{22}$$

Using the ladder operators, $\hat{a}_l(t)$ and $\hat{a}_l^\dagger(t)$, Eq. (20) can be represented in the simple form

$$\hat{I}_l = \hbar\Omega_l\left(\hat{a}_l^\dagger\hat{a}_l + \frac{1}{2}\right). \tag{23}$$

The eigenvalue of the invariant operator is

$$\lambda_{n,l} = \hbar\Omega_l\left(n_l + \frac{1}{2}\right), \tag{24}$$

where $n_l = 0, 1, 2, \cdots$ is the eigenvalue of $\hat{a}_l^\dagger(t)\hat{a}_l(t)$. Using ladder operators, $\hat{q}_l$ and $\hat{p}_l$ can be represented as

$$\hat{q}_l = \sqrt{\frac{\hbar}{2\epsilon\Omega_l}}\exp\left(-\frac{\sigma}{2\epsilon}t\right)\left[\hat{a}_l^\dagger(t) + \hat{a}_l(t)\right], \tag{25}$$

$$\hat{p}_l = i\sqrt{\frac{\epsilon\Omega_l\hbar}{2}}\exp\left(\frac{\sigma}{2\epsilon}t\right)\left\{\left(1 + i\frac{\sigma}{2\Omega_l\epsilon}\right)\hat{a}_l^\dagger(t) - \left(1 - i\frac{\sigma}{2\Omega_l\epsilon}\right)\hat{a}_l(t)\right\}. \tag{26}$$

Inserting Eq. (25) into Eq. (4), after performing some algebra, we obtain the following vector potential form:

$$\mathbf{A}(\mathbf{r}, t) = \sqrt{\frac{\hbar}{2\epsilon}}\exp\left(-\frac{\sigma}{2\epsilon}t\right)\sum_l \mathbf{u}_l(\mathbf{r})\frac{1}{\sqrt{\Omega_l}}\left[\hat{a}_l(0)\exp\left(-i\Omega_l t\right) + \hat{a}_l^\dagger(0)\exp\left(i\Omega_l t\right)\right]. \tag{27}$$

The above equation says that $\mathbf{A}(\mathbf{r},t)$ decays exponentially. The $\mathbf{E}$ and $\mathbf{B}$ field operator may be derived by inserting the above equation into Eqs. (10) and (11), respectively. Then we can confirm that the amplitude of electric and magnetic fields obviously both decay proportionally to $\exp[-\sigma t/(2\epsilon)]$.

We can obtain the ground eigenstate, $\langle \hat{q}_l | u_{0,l}(t)\rangle$, of the invariant operator from the fact that $\hat{a}_l(t)$ acting on the ground eigenstate gives zero. The $n$th eigenstate is obtained by acting with $\hat{a}_l^\dagger(t)$ on the ground eigenstate $n_l$ times. After performing normalization, the $n$th eigenstate of the invariant operator in $\hat{q}$-space is given by

$$\begin{aligned}\langle \hat{q}_l | u_{n,l}(t)\rangle &= \frac{1}{\sqrt{2^{n_l} n_l!}} \left(\frac{\epsilon\Omega_l}{\pi\hbar}\right)^{1/4} H_{n_l}\left[\sqrt{\frac{\epsilon\Omega_l}{\hbar}} \exp\left(\frac{\sigma}{2\epsilon}t\right)\hat{q}_l\right] \\ &\times \exp\left[\frac{\sigma}{4\epsilon}t - \frac{1}{2\hbar}\exp\left(\frac{\sigma}{\epsilon}t\right)\left(\epsilon\Omega_l + \frac{i\sigma}{2}\right)\hat{q}_l^2\right], \end{aligned} \tag{28}$$

where $H_{n_l}$ is the $n_l$th order Hermite polynomial.

## V. THE WAVE FUNCTION AND THE QUANTUM MECHANICAL ENERGY

The wave function satisfying the Schrödinger equation with the Hamiltonian Eq. (17), differs only by a time-dependent phase factor, $\exp[i\Lambda_{n,l}(t)]$, from the eigenstate of the invariant operator [15]:

$$\langle \hat{q}_l | \psi_{n,l}(t)\rangle = \langle \hat{q}_l | u_{n,l}(t)\rangle \exp[i\Lambda_{n,l}(t)]. \tag{29}$$

By inserting Eqs. (17) and (29) into the Schrödinger equation we obtain

$$\Lambda_{n,l}(t) = -\left(n_l + \frac{1}{2}\right)\Omega_l t. \tag{30}$$

Then the wave function given in Eq. (29) will coincide with that in Ref. [4]; it not only constitutes an orthonormal system but also has the property of completeness. Equation (29) may be used to calculate various expectation values for the quantum variables in the number state. For example, we can calculate the fluctuation of $\hat{q}_l$ and $\hat{p}_l$ as follows:

$$\begin{aligned}\Delta\hat{q}_l &= (\langle \psi_{n,l}(t)|\hat{q}_l^2|\psi_{n,l}(t)\rangle - \langle \psi_{n,l}(t)|\hat{q}_l|\psi_{n,l}(t)\rangle^2)^{1/2} \\ &= \left[\left(n_l + \frac{1}{2}\right)\frac{\hbar}{\epsilon\Omega_l}\exp\left(-\frac{\sigma}{\epsilon}t\right)\right]^{1/2}, \end{aligned} \tag{31}$$

$$\begin{aligned}\Delta\hat{p}_l &= (\langle \psi_{n,l}(t)|\hat{p}_l^2(t)|\psi_{n,l}(t)\rangle - \langle \psi_{n,l}(t)|\hat{p}_l|\psi_{n,l}(t)\rangle^2)^{1/2} \\ &= \left[\left(n_l + \frac{1}{2}\right)\hbar\epsilon\frac{\omega_l^2}{\Omega_l}\exp\left(\frac{\sigma}{\epsilon}t\right)\right]^{1/2}. \end{aligned} \tag{32}$$

In the above calculations, we used Eqs. (25) and (26) and the conventional relations given by

$$\hat{a}_l(t)|\psi_{n,l}(t)\rangle = \sqrt{n_l}\ \exp(-i\Omega_l t)|\psi_{n-1,l}(t)\rangle, \tag{33}$$

$$\hat{a}_l^\dagger(t)|\psi_{n,l}(t)\rangle = \sqrt{n_l+1}\ \exp(i\Omega_l t)|\psi_{n+1,l}(t)\rangle. \tag{34}$$

From Eqs. (31) and (32), we can easily obtain the uncertainty relation of $\hat{q}_l$ and $\hat{p}_l$ with respect to $|\psi_{n,l}(t)\rangle$:

$$\Delta\hat{q}_l\Delta\hat{p}_l = \left(n_l + \frac{1}{2}\right)\hbar\frac{\omega_l}{\Omega_l}. \tag{35}$$

The solutions of Eq. (31) approach zero as time approaches infinity, but never disappear, so that the uncertainty, Eq. (35), is always larger than $\hbar/2$. For the same reason, the zero point energy of the system does not vanish as time goes by, even though it becomes infinitesimally small. Thus, the uncertainty principle relation always holds as expected. We can see that the uncertainty of the $n$th eigenstate does not depend on time and that its value becomes large as the conductivity increases.

From the energy expression, Eq. (12), we find that the individual mechanical energy operator can be represented as

$$E_l = \frac{1}{2}\left[\exp\left(-\frac{2\sigma}{\epsilon}t\right)\frac{\hat{p}_l^2}{\epsilon} + \epsilon\omega_l^2\hat{q}_l^2\right]. \tag{36}$$

The expectation value of the quantum kinetic and potential energies with respect to $|\psi_{n,l}(t)\rangle$ can be derived from

$$T_{n,l} = \frac{1}{2\epsilon}\exp\left(-\frac{2\sigma}{\epsilon}t\right)\langle\psi_{n,l}(t)|\hat{p}_l^2|\psi_{n,l}(t)\rangle\,, \tag{37}$$

$$V_{n,l} = \frac{1}{2}\epsilon\omega_l^2\langle\psi_{n,l}(t)|\hat{q}_l^2|\psi_{n,l}(t)\rangle\,. \tag{38}$$

Using Eqs. (25), (26), and (29), the above two expressions can be easily calculated:

$$T_{n,l} = V_{n,l} = \left(n_l + \frac{1}{2}\right)\exp\left(-\frac{\sigma}{\epsilon}t\right)\hbar\frac{\omega_l^2}{2\Omega_l}\,. \tag{39}$$

Thus, we can confirm that the quantum kinetic energy and quantum potential energy equally contribute to the quantum mechanical energy. In the case where $\sigma = 0$, the above two quantum energies exactly reduce to that of the well known ordinary simple harmonic oscillator:

$$T_{n,l} = V_{n,l} = \frac{1}{2}\hbar\omega_l\left(n_l + \frac{1}{2}\right)\,. \tag{40}$$

By adding $T_{n,l}$ and $V_{n,l}$, we can easily identify the expectation value of $E_l$ with respect to $|\psi_{n,l}(t)\rangle$:

$$E_{n,l} = \left(n_l + \frac{1}{2}\right)\exp\left(-\frac{\sigma}{\epsilon}t\right)\hbar\frac{\omega_l^2}{\Omega_l}. \tag{41}$$

We can also obtain a similar to Eq. (41) type of energy expectation values for propagating light under periodic boundary conditions having discrete or continuous modes. The total quantum mechanical energy of the $n$th state is the sum of the individual energy modes, $E_n = \sum_l E_{n,l}$.

The classical kinetic and potential energies, Eqs. (13) and (14), oscillate with time. However, the quantum kinetic and potential energies in a number state, Eq. (39), obviously do not oscillate, regardless of the existence of conductivity, $\sigma$, which is responsible for dissipation, because the quantum cases are related to probability. For the same reason Eq. (41) does not oscillate.

By comparing Eq. (41) with Eq. (16) we can confirm that the decrease in the quantum and classical energy both occur in the same fashion, as the amplitude of the oscillation decreases exponentially with time. For larger conductivity the decrease is faster. The main difference between the quantum and classical energy result is that the former is quantized while the later is not quantized.

## VI. SUMMARY AND DISCUSSION

The wave function, $|\psi_{n,l}(t)\rangle$, is the state in which $n$ photons oscillate. It is also called the photon number state or Fock state, because it is simultaneously the eigenstate of the number operator $n_l$. We can say that the number operator $n_l$ is observable, since the photon number state satisfies orthonormal conditions and constitutes a basis in state space.

The number of photons does not vary with time in this quantized system. However, from Eq. (41), even if a group of photons which satisfies the damped wave equation, i.e. Eq. (3), have energy of $(n_l + 1/2)\hbar\omega_l^2/\Omega_l$ at $t = 0$, the energy decreases exponentially with the time, depending on the conductivity $\sigma$ and permittivity $\epsilon$ of the media. Furthermore the zero point energy, even though it is $\hbar\omega_l^2/(2\Omega_l)$ at the initial time, also decreases with time in the same way. The energy decrease is continuous. Note that it is not discrete. Thus, we can see that the energy of a single-photon is not always $\hbar\omega_l$, except for the case $\sigma = 0$. From Eq. (41) we can see that the quantum mechanical energy of a single-photon is

$$E_l(\text{single-photon}) = \hbar\frac{\omega_l^2}{\Omega_l}\exp\Big(-\frac{\sigma}{\epsilon}t\Big). \tag{42}$$

During dissipation the moment that the energy of a single-photon passing through the value $\hbar\omega_l$ is $t = (\epsilon/\sigma)\ln[\omega_l/\Omega_l]$. This seems to contradict previous knowledge, that the energy of a single-photon only depends on the frequency $\omega_l$ and it's value is always $\hbar\omega_l$, for a reason which has not yet been exactly understood [16]. However, we now find that this concept can be adopted only in non-dissipative system. Thus we conclude that the energy of light in a conductive media is evidently quantized to a specific number of photons, but the individual energy of a single-photon varies continuously with time although the frequency is not altered. Owing to the development of modern technology, the dissipation of a single-photon may be confirmed by careful experiments using a single-photon detector [17] in the far-infrared range. We can think that the energy of a single-photon depends not only on the frequency but also on the amplitude of the field. We confirm that the amplitudes of the electric and magnetic fields exponentially and continuously decrease in proportion to

$\exp[-\sigma t/(2\epsilon)]$ as time goes by. Maybe other dissipative and(or) time-dependent quantum systems also have properties similar to this [18–21]. The energy lost by a damped photon is used to generate heat in the media, according to the Joule heating law [22, 23]. In other words the lost energy is converted to heat energy. From Eq. (35) we can see that the uncertainty of the $n_l$th eigenstate does not depend on time, and its value becomes large as the conductivity increases.

## Acknowledgments

The author wishes to thank Professor K.H. Yeon in Chungbuk National University (Korea) for offering to him the issue of the quantization of the dissipative light.

## References

* Electronic address: choiardor@hanmail.net
[1] W. H. Louisell, *Quantum Statistical Properties of Radiation* (John Wiley and Sons, New York, 1973), Chaps. 3. 4. 6 and 7.
[2] M. O. Scully and M. S. Zubairy, *Quantum Optics* (Cambridge Univ. Press, Cambridge 1997), Chaps. 8 and 9.
[3] I. A. Pedrosa, Phys. Rev. A. **55**, 3219 (1997).
[4] I. A. Pedrosa G. P. Serra and I. Guedes, Phys. Rev. A. **56**, 4300 (1997).
[5] J. Y. Ji, J. K. Kim and S. P. Kim, Phys. Rev. A. **51**, 4268 (1995).
[6] J. Y. Ji, J. H. Kim, S. P. Kim and K. S. Soh, Phys. Rev. A. **52**, 3352 (1995).
[7] D. Y. Song, Phys. Rev. A. **62**, 014103 (2000).
[8] M. Maamache and H. Choutri, J. Phys. A: Math. Gen. **33**, 6203 (2000).
[9] H. R. Lewis Jr., Phys. Rev. Lett. **18**, 510 (1967).
[10] M. H. Nayfeh and M. K. Brussel, *Electricity and Magnetism* (John Wiley and Sons, New York, 1985), p.471.
[11] J. G. Gho, *Mechanics* (Cheong Moon Kak, Seoul, 1987), p.114.
[12] K. H. Yeon, C. I. Um and T. F. George, Phys. Rev. A **36**, 5287 (1987).
[13] H. Feshbach and Y. Tikochinsky, *A Festschrift for I. I. Rabi* (Trans. New York Acad. Sci. Ser. 2), **38**, 44 (1977).
[14] K. H. Yeon, K. K. Lee, C. I. Um T. F. George and L. N. Pandey, Phys. Rev. A **48**, 2716 (1993).
[15] H. R. Lewis Jr. and W. B. R. Riesenfeld, J. Math. Phys. **10**, 1458 (1969).
[16] K. Stowe, *Introduction to Statistical Mechanics and Thermodynamics* (John Wiley and Sons, New York, 1984), p.445
[17] S. Komiyama, O. Astafiev, V. Antonov, T. Kutsuwa and H. Hiral, Nature **403**, 405 (2000).
[18] J. R. Choi and S. Zhang, J. Phys. A: Math. Gen. **35**, 2845 (2002).
[19] J. R. Choi, Physica A **310**, 109 (2002).
[20] C. I. Um, J. R. Choi and K. H. Yeon, J. Korean Phys. Soc. **38**, 447 (2001).
[21] C. I. Um, J. R. Choi and K. H. Yeon, J. Korean Phys. Soc. **38**, 452 (2001).
[22] D. J. Griffiths, *Introduction to Electrodynamics* 2nd Ed. (Prentice Hall, Englewood Cliffs, 1989) pp. 374–375.
[23] J. R. Reitz, F. I. Milford and R. W. Christy, *Foundations of Electromagnetic Theory* 4th Ed. (Addison-Wesley, Massachusetts, 1993). pp. 465–469.