Natural Frequencies of a Lightly Loaded Hollow Biconic Beam

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Rapid evaluations of the frequencies of small-amplitude transverse vibrations of a loaded hollow biconic beam with a linearly varying wall thickness are determined by means of an analytically approximate approach. The natural frequencies are treated as those of an unloaded double-tapered circular beam perturbed by the mass of a light load. It is found that the first-order coefficient of the frequency-parameter change is independent of mode frequency.

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The design and analysis of structures to resist dynamic loading requires a knowledge of their natural vibration frequencies. Recently Lee and Chao presented a study of the exact solutions of the small-amplitude, undamped transverse vibration of a thin-walled biconic beam loaded with a central mass [1]. The natural frequencies and modeshapes of the beam system were determined for various values of the ratio of the central mass to the beam mass. This brief note deals with the same vibrational beam system, extending the previous investigation to search for a rapid approach that will result in reasonably acceptable eigenvalues for the corresponding frequency equation.

A hollow biconic beam is considered to possess a linearly varying wall thickness and is attached to a concentrated mass at the axial center. All the structural conditions and physical assumptions have been given previously [1] and an illustration of the beam system is shown in Fig. 1. Let the dimensionless frequency parameter for the free vibration, ©, be defined as $2(2½!^2=Ek^4)^{1=4}R^{1=2}$, where ½ is the mass density of the beam, ! denotes the natural frequency of the vibration, E stands for Young's modulus of elasticity, and k represents the ratio of the beam radius R to the beam length L. For appropriate boundary conditions, the dimensionless frequency parameter of the beam is found to satisfy

$$12I_2(©)J_2(©) + ©\left[I_2(©)J_1(©) + J_2(©)I_1(©)\right]¹ = 0; \qquad (1)$$

where ¹ is the mass ratio, defined as the ratio of central mass to beam mass, and J_n and I_n are, respectively, the ordinary and modified Bessel functions of the first kind of order n. The first five roots of Eq. (1) have been presented in an earlier paper for a wide spectrum of various mass ratios. The numerical results therein imply that the greater the load mass the lower the natural frequency [1].

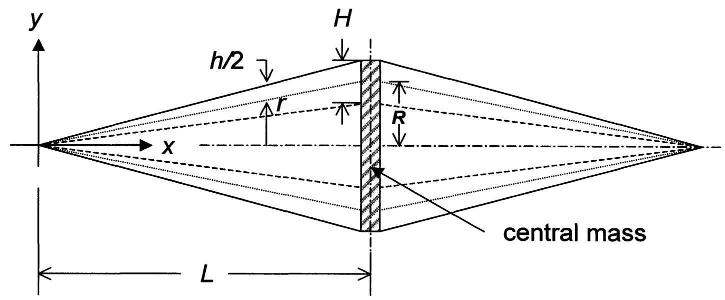

FIG. 1. Structure of the loaded hollow beam system. Shown in this figure is a thin-walled biconic beam with a linearly varying wall thickness.

It is of interest to see how the frequencies of an unloaded beam are altered by a light central mass as the source of perturbation. As pointed out previously, the frequency equation – Eq. (1) – in the limiting case of a zero mass ratio (¹ = 0), reduces to the simple form

$$J_2(©_0) = 0; \tag{2}$$

where the roots $©_0$, known as the zeros of the Bessel function J_2, can be found in most mathematics references [2]. The connection with $©_0$ allows the frequency parameter © to be expressed as a power series function of ¹ in the following form:

$$© = ©_0 + ¢©(¹) = ©_0 + ©_1(¹) + ©_2(¹^2) + ©_3(¹^3) + ::: ; \tag{3}$$

where $©_n$, n = 1, 2, 3, :::, are small compared with the unaltered frequency parameter $©_0$.

Ignoring the second-order and higher-order corrections, we write © ¼ $©_0 + ©_1$. After adopting the Taylor series expansions of the Bessel functions in Eq. (1) to the first order about $©_0$, the frequency equation, for the free vibration of a thin-walled beam with a linearly varying wall thickness, becomes

$$12 £ I_2(©_0) + ©_1 I^0_2(©_0) ¤ £ 0 + ©_1 J^0_2(©_0) ¤ + ©_0 I_2(©_0) J_1(©_0) ¹ = 0; \tag{4}$$

where the prime denotes differentiation with respect to the argument ©. Note that the last term on the left-hand side of the above equation is a result of neglecting the second and higher powers of ¹ or $©_1$ in the original expansion. We thus obtain

$$©_1 = ¡ (¹=12) £ J_1(©_0)=J^0_2(©_0) ¤ ©_0: \tag{5}$$

The identity properties of the Bessel functions and their derivatives permit the first-order perturbation in the frequency parameter to become

$$©_1 = ¡ (¹=12)©_0: \tag{6}$$

This indicates that the linear term of the frequency parameter is negative and depends merely on the mass ratio. Let us consider the coefficient of a frequency-parameter change defined as $(\Delta\Phi/\Phi_0)/\mu$. It is clear that the linear coefficient of the frequency-parameter change, given by $(\Phi_1/\Phi_0)/\mu$, is not involved with the frequency itself.

Eq. (6) yields an analytical expression for Φ, to first order in the mass ratio,

$$\Phi^{(m)} = \Phi_0^{(m)} + \Phi_1^{(m)} = \Phi_0^{(m)} - (\Phi_0^{(m)}/12)\mu; \tag{7}$$

where m stands for the modal number of the vibration and $\Phi_0^{(m)}$ is the m-th zero of J_2. The above equation can be evaluated directly by finding the zeros of J_2 from an existing mathematical table. The explicit forms for the first three modes are:

$$\Phi^{(1)} = 5.1356 - 0.4280\mu; \tag{8a}$$

$$\Phi^{(2)} = 8.4172 - 0.7014\mu; \tag{8b}$$

and

$$\Phi^{(3)} = 11.6198 - 0.9683\mu. \tag{8c}$$

To demonstrate how the simplified form, $\Phi \approx \Phi_0 + \Phi_1$, can be used as an estimate to facilitate the evaluation of the frequency parameter, consider the lightly loaded case of a beam with a linearly varying wall thickness. Table I illustrates some approximations for four distinct values of μ in comparison with the corresponding roots of Eq. (1). It is evident that the first-order approximations yield frequency parameters only $\sim$ 0.1% beyond the exact counterparts for small μ, up to $\sim$ 0.05. The numerical comparisons displayed in Table I also show that the explicit formulas expressed in Eqs. (8a)-(8c) yield more accurate results for lower modes and smaller mass ratios.

As the mass ratio becomes great enough to cause the linear term to dominate, Eq. (7) will become inapplicable. Nevertheless, instead of finding the numerical solutions of the frequency

TABLE I. Comparison of frequency parameters of the first three modes for the free vibration of a lightly loaded hollow beam with a linearly varying wall thickness.

Mass Ratio	Mode 1			Mode 2			Mode 3		
μ	Eq. (8)	Exact[a]	Error	Eq. (8)	Exact[a]	Error	Eq. (8)	Exact[a]	Error
0		5.1356			8.4172			11.6198	
0.01	5.13	5.13	0%	8.41	8.41	0%	11.61	11.61	0%
0.05	5.11	5.11	0%	8.38	8.37	0.1%	11.57	11.55	0.2%
0.1	5.09	5.08	0.2%	8.35	8.33	0.2%	11.52	11.50	0.2%
0.5	4.92	4.94	–0.4%	8.07	8.13	–0.7%	11.14	11.27	–1.2%

[a] Adapted from Table I, Ref. 1.

equation, entailing several multiplication terms of the Bessel functions, the analytical approximations give good estimates for lower modes and smaller load masses. In addition, the expansion can also provide a quick check against the numerical procedures in the computing of the roots. These advantages should warrant the value of the development of simplified polynomials of ^{1} for expressing the frequency parameter.

References

[1] W. Lee and H. Chao, J. Acoust. Soc. Am. **92**, 2260 (1992).

[2] See, for example, M. R. Spiegel, *Mathematical Handbook of Formulas and Tables* (McGraw-Hill, New York, 1990), p. 250.