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Statistical Entropy of a Rotating Cylindrical Black Hole

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By using the method of quantum statistics, we directly derive the partition function of bosonic and fermionic fields in a rotating cylindrical black hole. Then via the improved brick-wall method and the membrane model, we find that, if we choose the proper parameters, the entropy of the black hole is proportional to the area of the horizon. In our result, the stripped term and the divergent logarithmic term used in the original brick-wall method no longer exist. During the whole process, we do not make any approximations.

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I. Introduction

Since Bekenstein and Hawking put forward the idea that the entropy of a black hole is proportional to the area of its event horizon [1-3], the statistical origin of the black hole has been probed, and many ways of calculating the entropy emerged over time [4-9]. The most frequently used method among them is the brick-wall method advanced by G't Hooft [7]. This method is used to study the statistical property of a free scalar field in an asymptotically flat space-time in various spherical coordinates [10-12]. It was found that the general expression of the black hole's entropy consists of a term which is proportional to the area of its event horizon, as well as a divergent logarithmic term which is not proportional to the area of the event horizon. However, there are doubts. First, why is the entropy of the scalar or Dirac field outside the event horizon the entropy of the black hole? Second, the state density near the event horizon is divergent. Third, does the logarithmic term and L^3 term belong to the entropy of the black hole? Fourth, how does one calculate the entropy of a black hole described by non-spherical coordinates? The above mentioned problems in the original brick-wall method are unnatural and insurmountable.

It is well known that the entropy of a black hole is proportional to the area of its horizon and the existence of the horizon is a basic property of a black hole. It was proved that the existence of the horizon generally results in the Hawking effect [13]. Whether there is entropy of the black hole or not relates to the existence of the horizon [14]. A black hole's entropy has nothing to do with the radiation field outside the horizon. And the horizon only has the property of being a two-dimensional membrane in three-dimensional space. Does the number of quantum states of the two-dimensional membrane correspond to the entropy of the black hole? If it does, calculating the entropy of the membrane will be the key issue.

We derive the bosonic and fermionic partition functions for a rotating cylindrical black hole directly by using the quantum statistical method [15-17] and then obtain the integral expression of the system's entropy [15-20]. Then we use the membrane model t6o calculate the entropy. As a result, the left out term and the divergent logarithmic term in the original brick-wall method no lays exist. The problem of the divegence of the state density near the event horizon also no longer exists. We also consider the influence on the entropy of the spin degeneracy of particles. A neat way to study the entropy of various complicated balck holes is offered. In this article, we take the simplest functional form of the temperature $(C = \hbar = G = K_B = 1)$.

II. The rotating cylindrical balck hole

The line element of space-time for a rotating cylindrical black hole is given by [21]:

$$dS^2 = g_{tt}dt^2 + 2g_{t\varphi}dtd\varphi + g_{\varphi\varphi}d\varphi^2 + g_{rr}dr^2 + g_{zz}dz^2, \tag{2.1}$$

where

$$-g_{tt} = \alpha^2 r^2 - \frac{2(M+\ \ )}{\alpha r} + \frac{4Q^2}{\alpha^2 r^2}, \quad g_{t\varphi} = -\frac{8J}{3\alpha r}\left(1 - \frac{2Q^2}{(M+\ \ )\alpha r}\right),$$

$$g_{zz} = \alpha^2 r^2, \quad g_{\varphi\varphi} = r^2 + \frac{4(M-\ \ )}{\alpha^3 r}\left(1 - \frac{2Q^2}{(M+\ \ )\alpha r}\right),$$

$$g_{rr} = 1\Bigg/\left(\alpha^2 r^2 - \frac{2(3\ \ \ - M)}{\alpha r} + \frac{(3\ \ \ - M)4Q^2}{(\ \ + M)\alpha^2 r^2}\right),$$

where M, Q and J are the mass, charge and the angular momentum per unit height on the z axis. $\alpha^2 = -\frac{1}{3}\Lambda$, and Λ is a cosmological constant.

$$= \sqrt{M^2 - \frac{8J^2\alpha^2}{9}}, \tag{2.2}$$

The surface area per unit height on the z axis is

$$A_+ = 2\pi\alpha\left(r\sqrt{g_{\varphi\varphi}}\right)_{r=r_+}, \tag{2.3}$$

where r_+ is the location of the outer horizon; it satisfies

$$\Delta = \alpha^2 r^4 - \frac{2(3\ \ \ - M)}{\alpha} r + \frac{(3\ \ \ - M)4Q^2}{(M+\ \ )\alpha^2} = 0. \tag{2.4}$$

The Hawking radiation temperature of the black hole is

$$T_+ = \frac{1}{4\pi}\,\square\,\frac{1}{r\sqrt{g_{\varphi\varphi}}}\frac{d\Delta}{dr}\Bigg]_{r=r_+}. \tag{2.5}$$

In the view of Ref. [10, 22], the natural radiation temperature abserves by the observer at rest at an infinite distance is

$$T = \frac{T_+}{\sqrt{-g_{tt}}}, \tag{2.6}$$

where T_+ is the equilibrium temperature and

$$-g'_{tt} = \frac{g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2}{g_{\varphi\varphi}} = -\frac{\Delta}{g_{\varphi\varphi}}. \tag{2.7}$$

III. The bosonic entropy

For a bosonic gas, we calculate the partition functiion as follows:

$$\ln Z = -\sum_i g_i \ln(1 - e^{-\beta\varepsilon_i}). \tag{3.1}$$

In a unit volume, the number of quantum states with energy between ε and $\varepsilon + d\varepsilon$, or frequency between ε and $v + dv$, is

$$g(v)dv = j4\pi v^2 dv, \tag{3.2}$$

where j is the spin degeneracy of the particles. For the space-time (2.1), at a random point, the area of the two-dimensional cylindrical surface per unit height on the z axis is

$$A(r) = 2\pi\alpha r\sqrt{g_{\varphi\varphi}}. \tag{3.3}$$

Then, near the outside of the horizon at a random point r, the volume of the lamella per unit height on the z axis is

$$dV = A(r)\sqrt{g_{rr}}dr. \tag{3.4}$$

Then the partition function of the system at the lamella, with a random thickness at point r outside the horizon, is

$$\begin{aligned} \ln Z &= \int A(r)\sqrt{g_{rr}}dr \sum_i g_i \sum_{n=1} \frac{1}{n} e^{-n\beta\varepsilon_i} \\ &= j4\pi \int A(r)\sqrt{g_{rr}}dr \sum_{n=1} \frac{1}{n} \int_0^\infty e^{-\frac{nhv}{T}} v^2 dv \\ &= j\frac{1}{90}\pi^2 \int \frac{A(r)\sqrt{g_{rr}}dr}{\beta^3} = j\frac{\pi^3\alpha}{45} \int \frac{r\sqrt{g_{\varphi\varphi}g_{rr}}dr}{\beta^3}, \end{aligned} \tag{3.5}$$

where $\frac{1}{\beta} = T$. Using the relation between the entropy and the partition function:

$$S = \ln Z - \beta_0 \frac{\partial \ln Z}{\partial \beta_0}, \tag{3.6}$$

we get

$$S_b = j\frac{4\pi^3\alpha}{45}\frac{1}{\beta_0^3} \int \frac{r\sqrt{g_{\varphi\varphi}g_{rr}}dr}{(-g_{tt})^{3/2}} = j\frac{4\pi^3\alpha}{45}\frac{1}{\beta_0^3} \int \frac{r^2 g_{\varphi\varphi}^2 dr}{\Delta^2}, \tag{3.7}$$

where $\beta_0 = \frac{1}{T_+}$ and $\beta = \beta_0\sqrt{-g_{tt}}$. In the above integral (3.7), we take the integral region to be $[r_+ + \varsigma,\ r_+ + N\varsigma]$ where ς is a small non-negative quantity and N is a constant, larger than one. Then (3.7) ban be written as:

$$
\begin{aligned}
S_b &= j\frac{4\pi^3\alpha}{45}\frac{1}{\beta_0^3}\int_{r_+ + \varsigma}^{r_+ + N\varsigma} \frac{r^2 g_{\varphi\varphi}^2 dr}{\Delta^2} \\
&= j\frac{4\pi^3\alpha}{45}\frac{1}{\beta_0^3}\int_{r_+ + \varsigma}^{r_+ + N\varsigma} \frac{r^2 g_{\varphi\varphi}^2 dr}{(r-r_+)^2(r-r_2)^2(r-r_3)^2(r-r_4)^2} \\
&= j\frac{4\pi^3\alpha}{45}\frac{1}{\beta_0^3}\frac{r_+^6(M+\quad)^2}{(r_+-r_2)^2(r_+-r_3)^2(r_+-r_4)^2(3\quad -M)^2}\ \square\frac{N-1}{N\varsigma}\Big] \\
&\quad +j\frac{4\pi^3\alpha}{45}\frac{1}{\beta_0^3}f'(r_+)\ln N + j\frac{4\pi^3\alpha}{45}\frac{1}{\beta_0^3}F(r_+,\varsigma,N\varsigma),
\end{aligned}
\tag{3.8}
$$

where

$$
\begin{aligned}
f(r) &= \frac{r^2 g_{\varphi\varphi}^2}{(r-r_2)^2(r-r_3)^2(r-r_4)^2}, \\
F(r_+,\varsigma,N\varsigma) &= \int_{r_+ + \varsigma}^{r_+ + N\varsigma} = \left[\sum_{n=2}^{\infty}\frac{f^{(n)}(r_+)}{n!}(r-r_+)^{n-2}\right] dr, \\
f^{(n)}(r_+) &= \left.\frac{df}{dr}\right|_{r=r_+} \left. g_{\varphi\varphi}\right|_{r=r_+} = r_+^2\frac{M+\quad}{3\quad -M}.
\end{aligned}
\tag{3.9}
$$

From (3.17) in Ref. [7] we know that when $N\varsigma = L \gg r_+$, if we take

$$
\varsigma = \frac{T_+}{90},
\tag{3.10}
$$

we find that the entropy of black hole is proportional to the area of its horizon. In order to ensure that the radiation field and black hole are in stable equilibrium [23], the infrared cutoff should not be taken as $L \gg r_+$. Considering the above mentioned reason, for a rotating cylindrical black hole, we take as the ultraviolet cutoff

$$
\varsigma = \frac{T_+}{90}\sqrt{\frac{M+\quad}{3\quad -M}}\frac{N-1}{N},
\tag{3.11}
$$

and the infrared cutoff $N\varsigma$. The value of N should ensure that the radiation field and the black hole are in stable equilibrium. Then we have

$$
S_b = j\frac{1}{2}\pi\alpha(r\sqrt{g_{\varphi\varphi}})_{r=r_+} + \frac{4\pi^3\alpha}{45}\frac{1}{\beta_0^3}f'(r_+)\ln N + j\frac{4\pi^3\alpha}{45}\frac{1}{\beta_0^3}F(r_+,\varsigma,N\varsigma)
\tag{3.12}
$$

As $N \to 1$, $\varsigma \to 0$ and $N\varsigma \to 0$; that is, the ultraviolet cutoff and infrared cutoff both approach the outer horizon of the black hole; the entropy per unit height on the Z axis is

$$
S_b = j\frac{1}{4}A_+.
\tag{3.13}
$$

In our calculation, we make use of $\lim\limits_{N\to 1} F(r_+, N, \varsigma) \to 0$, where $A_+ = 2\pi\alpha(r\sqrt{g_{\varphi\varphi}})_{r=r_+}$ is the area of the horizon per unit height on the Z axis. When the ultraviolet cutoff and the infrared cutoff both approach the outer horizon of the black hole, the calculated entropy has nothing to do with the radiation field outside the black hole, it should be the black hole's entropy thus the entropy given by (3.13) is the cylindrical black hole's entropy per unit height on the Z axis.

IV. The fermionic entroph

For a fermionic gas, the partition function is

$$\ln Z = \sum_i g_i \ \ln(l + e^{-\beta\varepsilon_i}). \tag{4.1}$$

From (3.2), we obtain

$$\begin{aligned} \ln\, Z &= \int A(r)\sqrt{g_{rr}}dr \sum_i g_i \sum_{n=1} \frac{(-1)^{n-1}}{n}\, e^{-n\beta\varepsilon_i} \\ &= \omega 4\pi \int A(r)\sqrt{g_{rr}}dr \sum_{n=1} \frac{(-1)^{n-1}}{n} \int_0^\infty e^{-\frac{nhv}{T}} v^2 dv \\ &= \omega\frac{1}{90}\pi^2\frac{7}{8}\int \frac{A(r)\sqrt{g_{rr}}dr}{\beta^3} = \omega\frac{\pi^3\alpha}{45}\frac{7}{8}\int \frac{r\sqrt{g_{\varphi\varphi}g_{rr}}dr}{\beta^3}. \end{aligned} \tag{4.2}$$

Using (3.6), we can get the fermionic entropy of a rotating cylindrical black hole per unit height on the z axis.

$$\begin{aligned} S_f &= \omega\frac{4\pi^3\alpha}{45}\,\frac{7}{8}\,\frac{1}{\beta_0^3}\int \frac{r\sqrt{g_{\varphi\varphi}g_{zz}}dr}{(-g_{tt})^{3/2}} = \frac{4\pi^3\alpha}{45}\,\frac{7}{8}\,\frac{1}{\beta_0^3}\int \frac{r^2 g_{\varphi\varphi}^2 dr}{\Delta^2} \\ &= \omega\frac{4\pi^3\alpha}{45}\,\frac{7}{8}\,\frac{1}{\beta_0^3}\int_{r_+ + \varsigma}^{r_+ + N\varsigma} \frac{r^2 g_{\varphi\varphi}^2 dr}{(r-r_+)^2(r-r_2)^2(r-r_3)^2(r-r_4)^2} \\ &= \omega\frac{4\pi^3\alpha}{45}\,\frac{7}{8}\,\frac{1}{\beta_0^3}\,\frac{r_+^6(M + \quad)^2}{(r_+-r_2)^2(r_+-r_3)^2(r_+-r_4)^2(3 \quad - M)^2} \ {}^{\square}\frac{N-1}{N\varsigma}\Big] \\ &\quad +\omega\frac{4\pi^3\alpha}{45}\,\frac{7}{8}\,\frac{1}{\beta_0^3}f'(r_+)\ln N + \omega\frac{4\pi^3\alpha}{45}\,\frac{7}{8}\,\frac{1}{\beta_0^3}F(r_+, \varsigma, N\varsigma), \end{aligned} \tag{4.3}$$

where $f(r)$ and $F(r_+, \varsigma, N\varsigma)$ are given by (3.9), and ς is given by (3.11). However, when $N \to 1$, $\varsigma \to 1$ and $N\varsigma \to 0$, that is, the ultraviolet cutoff and infrared cutoff both approach the outer horizon of the black hole, the fermionic entropy is

$$S_f = \omega\frac{7}{8}\frac{1}{4}A_+, \tag{4.4}$$

where ω is the spin degeneracy of the fermionic particles. Now when we compare the bosonic entropy with the fermionic entropy of the rotating cylindrical black hole, we find that they have the same form, except for their coefficients, i.e., as $\omega = j$, the fermionic entropy is 7/8 times the bosoinc entropy.

V. Conclusion

In the above analysis, we derived the partition functions for various fields in a rotating cylindrical black hole spacetime directly by using the statistical method. We avoid the difficulty in volued in solving the wave equation. Since we use the improved brick-wall method, and the membrane model, to calculate the entropy of the various fields, the problem of the state density being divergent around the horizon no longer exists. In our calculation, as $N \to 1$, $\varsigma \to 0$ and $N\varsigma \to 0$, that is, the ultraviolet cutoff and infrared cutoff both approach the outer horizon of the black hole. However, from (3.13) and (4.4), we know that the calculated entropy has nothing to do with the radiation field, so the left-out term and the divergent logarithmic term in the original brick-wall method no longer exist. The obtained entropy is proportional to the area of the horizon, so it can be taken as black hole's entropy.

In above analysis, we find that by using statistical and membrane model methods, the doubthlness in the original brick wall method about the entropy of the scalar or Dirac field outside the event horizon being the entropy of the black hole no longer exists and the complicated approximations in the solution are avoided. We also consider the influence of the spin degeneracy of the particles on the entropy. To calculate the entropy in various space-times, we only need to change the red-shift factor, the other factors are the same. Especially for complicated space-times, we can directly derive the entropy of various quantum particles without solving a complicated wave equation. We offer a new nearer way to study the entropy of different kinds of complicated black holes.

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