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General-Linear-Acceleration Transformations of Spacetime, Jerks and Limiting 4-Dimensional Symmetry

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Based on the limiting 4-dimensional symmetry of the Lorentz and Poincaré groups, new general spacetime transformations for frames with arbitrary linear acceleration in any direction are obtained in a fairly simple form. A new feature in the metric tensors is the presence of jerk, the time derivative of acceleration. The Riemann curvature tensor of spacetime in these non-inertial frames vanishes. For one-dimensional motion, the general spacetime transformations are shown to form a group. The transformations of Wu, Møller, Poincaré and Lorentz are special limiting cases of this general-linear-acceleration transformation. The Planck constant $\hbar$ and the speed of light c are not invariant universal constants under such general spacetime transformations. The theory suggests that the electromagnetic coupling strength α_e = 1/137.035989 and a new quantum constant $J = 3.5177293 \times 10^{-38} g \cdot cm$, are truly universal constants in both inertial and non-inertial frames.

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I. Introduction

Almost all physical frames of reference in the universe are, strictly speaking, non-inertial because of the long range action of the 'gravitational force.' Thus it is desirable that the laws of physics and the universal and fundamental constants are understood or known not only in inertial frames but also in non-inertial frames [1, 2]. We know that inertial frames are idealizations or approximations of non-inertial frames in the limit of zero acceleration. Experiments have established that physical laws in inertial frames display Lorentz and Poincaré invariance. So it is natural and necessary to require that the laws of physics in non-inertial frames must display the 4-dimensional symmetry of the Lorentz and Poincaré groups in the limit of zero acceleration. Such a requirement is postulated as the principle of 'limiting 4-dimensional symmetry' [2]. But, the acceleration transformations in earlier works [1] do not naturally reduce to the Lorentz transformation in the limit of zero acceleration. It must be stressed that limiting 4-dimensional symmetry is simply the 4-dimensional symmetry of the Lorentz and the Poincaré groups applied to non-inertial frames in the limit of zero acceleration. In other words it is the first postulate (i.e., invariance of physical laws) rather than the second postulate of special relativity theory that is applied to non-inertial frames. The reason is obvious; because the speed of light cannot be a universal constant in non-inertial frames.

In previous works, the principle of limiting 4-dimensional symmetry has been used to derive a generalized Lorentz transformation, i.e., spacetime transformations for frames with constant-linear-acceleration [2]. These transformations form the Wu group, which includes the Møller group [3] and the Lorentz group as limiting cases. In this paper, the same limiting 4-dimensional symmetry is employed to generalize spacetime transformations for frames with arbitrary accelerations along a straight line. We show that the set of such general spacetime transformations form a group, called 'General Taiji (GT) group', which involves one arbitrary acceleration function $\alpha(w)$ and two parameters, i.e., the initial acceleration α_o and velocity β_o. 'Taiji' denotes, in ancient Chinese thought, the ultimate principle or the condition as it existed before the creation of the world. This appears to be a fitting name because the general taiji group includes the Wu and Møller groups for constant accelerations and the Lorentz and Poincaré groups for constant velocities as special limiting cases. The spacetime metric tensors of spacetime in these non-inertial frames are affected by the jerk, J_e, the third-order time derivative of coordinates. In engineering, jerks have been used in the design of servo tracking systems for fast moving objects and in robotics. In physics, it appears that jerks have not yet been used, except in the radiative reaction force. Now in the present theory local space and time axes, dx and dw, will no longer be orthogonal in the presence of the jerk, J_e. This property appears to be necessary for the set of GT transformations of spacetime to have a fairly simple form and to form a symmetry group.

We investigate the physics in general non-inertial frames with arbitrary accelerations along a straight line (relative to an inertial frame) by using a purely kinematic approach [4], independent of the gravitational field. We shall call such frames 'general-linear-acceleration' (GLA) frames. Physical results are thus obtained without using gravity or the approximate equivalence principle as a crutch.

Naturally, the "4-dimensional spacetime" (w, x, y, z) of non-inertial frames is more general than the Minkowski spacetime of inertial frames which is included as a special case when the acceleration approaches zero. To avoid confusion, let us call such a spacetime "general taiji (GT) spacetime." Since the constant c of the speed of light has no operational meaning in GLA frames, we will directly use w (with the dimension of length) instead of t (or ct) for the generalized evolution variable and call it the "GT time" (or simply time) for GLA frames [5].

II. Spacetime transformations with general-linear-accelerations in arbitrary directions

Based on the kinematical approach of limiting 4-dimensional symmetry, we first derive the transformation with an arbitrary velocity $\beta(w)$ in the +x-direction, and then we generalize $\beta(w)$ to an arbitrary direction. It is gratifying that the resultant transformations take a fairly simple form. Suppose that the frame F_c has both its velocity and acceleration directed along parallel x and x_I axes and that the origins of F_c and F_I coincide at the time $w = w_I = 0$. We obtain transformations for the constant-linear-acceleration frame $F_c(x)$ and an inertial frame $F_I(x_I)$: [2]

$$w_I = \gamma\beta x + \frac{\gamma\beta}{\alpha_o\gamma_o^2} + a_o, \quad x_I = \gamma x + \frac{\gamma}{\alpha_o\gamma_o^2} + b_o, \quad y_I = y, \quad z_I = z,$$
$$\beta = \alpha_o w + \beta_o, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \gamma_o = \frac{1}{\sqrt{1-\beta_o^2}}, \quad a_o = -\frac{\beta_o}{\alpha_o\gamma_o}, \quad b_o = -\frac{1}{\alpha_o\gamma_o}, \tag{1}$$

where the velocity $\beta = \beta(w)$ is a linear function time w. The result (1) is called the Wu transformation. It reduces to the Møller transformation when $\beta_o = 0$, provided a change of time variable ($w = (1/\alpha_o)\tanh(\alpha_o w*)$) is made [2]. Furthermore, in the limit of zero acceleration, $\alpha_o \to 0$, the Wu transformation (1) reduces to the 4-dimensional transformations (which form the

Lorentz group),

$$w_I = \gamma_o(w + \beta_o x), \quad x_I = \gamma_o(x + \beta_o w), \quad y_I = y, \quad z_I = z. \tag{2}$$

Thus, limiting 4-dimensional symmetry of the Lorentz and Poincaré invariance is satisfied. The differential form of the Wu transformation (1) for constant-linear-acceleration is

$$\begin{aligned} &dw_I = \gamma(W_c dw + \beta dx), \quad dx_I = \gamma(dx + \beta W_c dw), \\ &dy_I = dy, \quad dz_I = dz; \quad W_c = \gamma^2(\gamma_o^{-2} + \alpha_o x). \end{aligned} \tag{3}$$

The Møller transformation and its differential form are

$$\begin{aligned} &w_I = X \sin hQ, \quad x_I = X \cosh Q - \frac{1}{\alpha}, \quad y_I = y, \quad z_I = z; \\ &dw_I = dx \sinh Q + dw^* \alpha_o X \cosh Q, \quad dx_I = dx \cosh Q + dw^* \alpha_o X \sinh Q, \\ &X = x + \frac{1}{\alpha_o}, \quad Q = \alpha_o w^*, \quad \sinh Q = \gamma\beta, \quad \cosh Q = \gamma, \quad \beta = \tanh Q. \end{aligned} \tag{4}$$

Based on the differential froms in (3) and (4), we can consider the generalization of the Wu transformation (1) to a more general non-inertial frame $F(x)$ moving with an arbitrary velocity $\beta(w)$ or arbitrary acceleration $\alpha(w)$ along the x-axis [6],

$$\begin{aligned} &\beta(w) = \beta_1(w) + \beta_o, \quad \alpha(w) = \frac{d\beta(w)}{dw} = \frac{d\beta_1(w)}{dw}, \\ &\beta(0) = \beta_o, \quad \alpha(0) = \alpha_o. \end{aligned} \tag{5}$$

The last two initial conditions are related to the fact that the origins of $F(X)$ and $F_I(x_I)$ coincide at GT time $w = w_I = 0$. One of the simple generalizations of the constant-acceleration case (3) is to assume the following local relation for $F(x)$ and $F_I(x_I)$

$$\begin{aligned} &dw_I = \gamma(W_a dw + \beta dx), \quad dx_I = \gamma(dx + \beta W_b dw), \\ &dy_I = dy, \quad dz_I = dz, \\ &\gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \beta = \beta_1(w) + \beta_o, \quad \beta^2 < 1, \end{aligned} \tag{6}$$

where the two functions $W_a = W_a(w, x)$ and $W_b = W_b(w, x)$ are different in general. The velocity β is an arbitrary function of the GT time w and characterizes the general-linear-acceleration in the x-direction of a non-inertial frame F. This function must be given in order to specify the acceleration of a non-inertial frame. For example, if $\beta = J_{eo}w^2/2 + \alpha_o w + \beta_o$, then the non-inertial frame has a constant jerk J_{eo} and a variable acceleration $d\beta/dw = J_{eo}w + \alpha_o$, which is linear in time w. The local form (6) is a minimal generalization of the Wu transformations (3) for constant acceleration.

The heart of the problem is how to find a satisfactory local transformation which is integrable and satisfies limiting 4-dimensional symmetry. For such a generalization, simplicity and minimal departure from the Wu transformaton (1) are employed as guiding principles. The limiting 4-dimensional symmetry dictates that the two unknown functions $W_a(w, x)$ and $W_b(w, x)$ must satisfy the following two integrability conditions for the differential equations in (6):

$$\frac{\partial}{\partial x}(\gamma W_a) = \frac{\partial}{\partial w}(\gamma\beta), \quad \frac{\partial}{\partial x}(\gamma\beta W_b) = \frac{\partial}{\partial w}(\gamma). \tag{7}$$

Since γ and β are functions of w only, (7) implies the relations $\partial W_a/\partial x = \gamma^2\alpha$ and $\partial W_b/\partial x = \gamma^2\alpha$, which lead to the results

$$W_a = \gamma^2\alpha x + A(w), \quad W_b = \gamma^2\alpha x + B(w), \tag{8}$$

where the two functions $A(w)$ and $B(w)$ may not be the same. We substitute (8) into (6) and carry out the integrations, we obtain

$$w_I = \gamma\beta x + \int \gamma A(w) dw, \quad x_I = \gamma x + \int \gamma\beta B(w) dw. \tag{9}$$

Limiting 4-dimensional symmetry suggests that one of the simplest generalizations is to identify the two integrals in (9) with the two last terms in (1) in which the constant acceleration α_o is replaced by an acceleration function $\alpha(w)$ [6]:

$$\begin{aligned} &\int \gamma A(w) dw = \gamma\beta \frac{1}{\alpha(w)\gamma_o^2} + a_o, \\ &\int \gamma\beta B(w) dw = \gamma \frac{1}{\alpha(w)\gamma_o^2} + b_o. \end{aligned} \tag{10}$$

By differentiation of (10), we can determine the two functions $A(w)$ and $B(w)$,

$$A(w) = \frac{\gamma^2}{\gamma_o^2} - \frac{\beta J_e}{\alpha^2\gamma_o^2}, \quad B(w) = \frac{\gamma^2}{\gamma_o^2} - \frac{J_e}{\beta\alpha^2\gamma_o^2}, \quad J_e(w) = \frac{d\alpha}{dw} = \frac{d^2\beta}{dw^2}, \tag{11}$$

where $J_e(w)$ is the jerk, which is the third-order time derivative of the coordinate. From (9) and (11), we obtain a simple and general spacetime transformation for GLA frames

$$\begin{aligned} &w_I = \gamma\beta\left(x + \frac{1}{\alpha(w)\gamma_o^2}\right) - \frac{\beta_o}{\alpha_o\gamma_o}, \\ &x_I = \gamma\left(x + \frac{1}{\alpha(w)\gamma_o^2}\right) - \tfrac{1}{\alpha_o\gamma_o}, \\ &y_I = y, \quad z_I = z, \end{aligned} \tag{12}$$

where the two constants of integration $a_o = -\beta_o/(\alpha_o\gamma_o)$ and $b_o = -1/(\alpha_o\gamma_o)$ are determined by the limiting 4-dimensional symmetry as $\alpha(w) = \alpha_o \to 0$. The relations in (12) may be termed "general taiji (GT) transformation" for a frame $F(x)$ with a general linear acceleration $\alpha(w)$ in the x direction.

The GT transformation in equation (12) can be generalized to the case in which the general velocity $\beta(w)$ or the linear acceleration is in an arbitrary direction. In this case, $\beta(w)$, β_o and

$\alpha(w)$ are in the same fixed direction, so that we still have the general-linear-acceleration. The resultant GT spacetime transformations take the following form

$$w_I = \gamma\left(\frac{\beta}{\alpha(w)\gamma_o^2} + \beta\cdot\mathbf{r}\right) - \frac{\beta_o}{\alpha_o\gamma_o},$$
$$\mathbf{r}_I = \mathbf{r} + (\gamma-1)(\beta\cdot\mathbf{r})\frac{\beta}{\beta^2} + \left(\frac{\gamma}{\alpha(w)} - \frac{\gamma_o}{\alpha_o}\right)\frac{\beta}{\beta\gamma_o^2}. \tag{12a}$$

From the GT transformation (12), we can obtain a simple GT transformations for the differentials dx^μ and dx_I^μ,

$$dw_I = \gamma(W_a dw + \beta dx), \quad dx_I = \gamma(dx + \beta W_b dw),$$
$$dy_I = dy, \quad dz_I = dz,$$
$$W_a = \gamma^2\left(\alpha x + \frac{1}{\gamma_o^2}\right) - \frac{\beta J_e(w)}{\alpha^2\gamma_o^2} > 0, \quad W_b = \gamma^2\left(\alpha x + \frac{1}{\gamma_o^2}\right) - \frac{J_e(w)}{\beta\alpha^2\gamma_o^2} > 0. \tag{13}$$

The invariant interval ds^2 in GLA frames can be obtained from (13),

$$ds^2 = dw_I^2 - dx_I^2 - dy_I^2 - dz_I^2 = g_{\mu\nu}dx^\mu dx^\nu$$
$$= W^2 dw^2 + 2U dw dx - dx^2 - dy^2 - dz^2,$$
$$W^2(w,x) = \left.\gamma^2\left(\alpha x + \frac{1}{\gamma_o^2}\right)\right]^2 - \left.\frac{J_e(w}{\alpha^2\gamma_o^2}\right]^2 > 0, \quad U = \frac{J_e(w)}{\alpha^2\gamma_o^2}. \tag{14}$$

When the jerk $J_e(w) = d\alpha(w)/dw$ vanishes, we have $\alpha(w) = \alpha_o$ and one can see that the GT transformation (12) reduces to the Wu transformation (1) for a constant-linear-acceleration frame $F_c(x)$, in which the time axis is everywhere orthogonal to the spatial coordinate curves. From (14), we have the following fundamental spacetime metric tensors for a general non-inertial frame,

$$g_{00} = W^2, \quad g_{01} = g_{10} = U, \quad g_{11} = g_{22} = g_{33} = -1,$$
$$g^{00} = \frac{1}{W^2 + U^2}, \quad g^{01} = g^{10} = \frac{U}{W^2 + U^2}, \quad g^{11} = \frac{-W^2}{W^2 + U^2}, \quad g^{22} = g^{33} = -1, \tag{15}$$

where $g^{\alpha\gamma}g_{\gamma\beta} = \delta^\alpha_\beta$ and all other components vanish.

The coordinates x^μ specified by the metric tensor in (15) for a non-inertial frame may be called "general taiji spacetime." They are the preferred coordinates for the general taiji transformation with limiting 4-dimensional symmetry. Other choices of coordinates will not satisfy the limiting 4-dimensional symmetry. Thus, the present theory of spacetime for general non-inertial frames is not a general covariant theory, in contrast to the general theory of relativity.

One can generalize the GT transformation (12) to include constant translations $x_o^\mu = (w_o, x_o, y_o, z_o)$,

$$w_I = \gamma\beta\left(x + \frac{1}{\alpha(w)\gamma_o^2}\right) - \frac{\beta_o}{\alpha_o\gamma_o} + w_o,$$
$$x_I = \gamma\left(x + \frac{1}{\alpha(w)\gamma_o^2}\right) - \tfrac{1}{\alpha_o\gamma_o} + x_o,$$
$$y_I = y + y_o, \quad z_I = z + z_o. \tag{16}$$

In this case, when the jerk $J_e(w)$ vanishes, we have $\alpha(w) = \alpha_o$ and (16) becomes a generalized Wu transformation with constant translations. Furthermore, in the limit of zero acceleration $\alpha(w) = \alpha_o \to 0$, the GT transformation (16) reduces to the Poincaré transformation. We shall not discuss (16) further in the present paper.

Suppose the arbitrary acceleration $\alpha = \alpha(w)$ can be expressed in terms of quantities in the inertial frame $\alpha(w) = \alpha_I(w_I, x_I)$, the inverse of the GT transformation (12) is

$$\begin{aligned} \beta &= \left[\frac{w_I + \beta_o/(\alpha_o\gamma_o)}{x_I + 1/(\alpha_o\gamma_o)}\right], \\ x &= \sqrt{\left(x_I + \frac{1}{\gamma_o\alpha_o}\right)^2 - \left(w_I + \tfrac{\beta_o}{\alpha_o\gamma_o}\right)^2} - \frac{1}{\alpha\gamma_o^2}, \\ y &= y_I, \qquad z = z_I. \end{aligned} \tag{17}$$

If a specific function for $\beta(w)$ is given and one can solve for the time w in terms of β, then the first equation in (17) can be written in the form $w = w(\beta)$. For example, the simplest generalization of constant-linear acceleration is the case with a constant jerk, $J_{eo} = constant$ or $\beta = J_{eo}w^2/2 + \alpha_o w + \beta_o$. When w is positive, we have $w = (1/J_{eo})[-\alpha_o + \sqrt{\alpha_o^2 + 2J_{eo}(\beta - \beta_o)}]$, so that we have the transformation for time, $w = w(w_I, x_I)$.

The inverse GT transformation with the velocity $\beta(w)$ in an arbitrary direction can be derived from (12a):

$$\begin{aligned} \beta(w) &= \left[\frac{w_I + \beta_o/(\alpha_o\gamma_o)}{\mathbf{n}\cdot\mathbf{r}_I + 1/(\alpha_o\gamma_o)}\right], \\ \mathbf{r} &= \mathbf{r}_I + \left(\frac{\gamma_o}{\alpha_o} - \frac{1}{\alpha(w)}\right)\frac{\mathbf{n}}{\gamma_o^2} - \frac{(\gamma-1)\mathbf{n}}{\gamma\beta}\left(w_I + \frac{\beta_o}{\alpha_o\gamma_o}\right), \\ \sigma(w) &= \alpha_I(w_I, x_I), \quad \mathbf{n} = \frac{\boldsymbol{\beta}}{\beta} = \frac{\boldsymbol{\beta}}{\beta_o}, \end{aligned} \tag{17a}$$

provided the general acceleration $\alpha = \alpha(w)$ can be expressed in terms of quantities measured in the inertial frame, $\alpha(w) = \alpha_I(w_I, \mathbf{r}_I)$. For $\boldsymbol{\beta}(w) = \mathbf{n}\beta$ in an arbitrary direction, the invariant interval $ds^2 = g_{\mu\nu}dx^\mu dx^\nu = g_{00}dw^2 + 2Udw\mathbf{n}\cdot d\mathbf{r} - d\mathbf{r}^2$ resembles to that in (14), exccept that $g_{00} = W(w, \mathbf{r})^2$ is now given by

$$W^2(w, \mathbf{r}) = \left[\gamma^2\left(\alpha(w)(\mathbf{n}\cdot\mathbf{r}) + \frac{1}{\gamma_o^2}\right)\right]^2 - \left[\frac{J_e(w)}{\alpha^2(w)\gamma_o^2}\right]^2.$$

III. The non-constant speed of light and the velocity-addition law

Just as in an inertial frame, the law for the propagation of light in a non-inertial frame is described by the invariant equation $ds = 0$, where

$$\begin{aligned} ds^2 &= W^2dw^2 + 2Udwdx - dx^2 - dy^2 - dz^2 \\ &= \left(Wdw + \tfrac{U}{W}dx\right)^2 - \left[\left(\frac{U^2}{W^2} + 1\right)dx^2 + dy^2 + dz^2\right]. \end{aligned} \tag{18}$$

The distance in terms of the space coordinate elements is given by [7]

$$dr_s^2 = \left[g_{\mu\nu} - \frac{g_{0\mu}g_{0\nu}}{g_{00}} \right] dx^\mu dx^\nu, \tag{19}$$

which is the same as the last three terms in (18). In order to see the property of light in a GLA frame, let us consider some specific and simple cases. Suppose a light signal moves along the x-axis, i.e., $dy = dz = 0$, the speed of light $\beta_L$is found to be

$$\beta_{Lx} = \frac{dx}{dw} = \gamma^2(\alpha x + \gamma_o^{-2}) + \frac{J_e(w)}{\alpha^2\gamma_o^2}, \tag{20}$$

which is certainly different from the speed of light $\beta_L = 1$ (derived from $ds^2 = dw_I^2 - dx_I^2 = 0$) in an inertial frame. If the signal moves in the y-direction, i.e., $dx = dz = 0$, Eq. (18) leads to the speed of light β_{Ly}

$$\beta_{Ly} = \frac{dy}{dw} = \sqrt{\left[\gamma^2 \left(\alpha x + \frac{1}{\gamma_o^2}\right)\right]^2 - \left[\frac{J_e(w}{\alpha^2\gamma_o^2}\right]^2}. \tag{21}$$

The law of velocity addition can be obtained from (13),

$$\begin{aligned} \frac{dx_I}{dw_I} &= \frac{dx/dw + \beta W_b}{W_a + \beta dx/dw}, \\ \frac{dy_I}{dw_I} &= \frac{dy/dw}{\gamma(W_a + \beta dx/dw)}, \quad \frac{dz_I}{dw_I} = \frac{dz/dw}{\gamma(W_a + \beta dx/dw)}, \end{aligned} \tag{22}$$

where W_a and W_b are given in (13). The results in (20) and (21) can also be obtained from the law of velocity addition in equation (22).

IV. The general taiji group

One can verify that, in the limit $\alpha(w) = \alpha_o \to 0$ and $\beta_o \to 0$, the GT transformation (12) reduces to the identity transformation. The inverse transformation of (12) is given by (17). To show other group properties of the GT transformation (12) for an arbitrary acceleration along the x-axis, let us consider two other GLA frames F' and F'', which are respectively characterized by arbitrary velocities $\beta'(w')$, $\beta''(w'')$, initial accelerations α'_o, α''_o, and initial velocities β'_o, β''_o. Thus we have the GT transformations among F_I, F, F' and F'',

$$\begin{aligned} w_I &= \gamma\beta\left(x + \frac{1}{\alpha(w)\gamma_o^2}\right) - \frac{\beta_o}{\alpha_o\gamma_o} = \gamma'\beta'\left(x' + \frac{1}{\alpha'(w')\gamma'^2_o}\right) - \frac{\beta'_o}{\alpha'_o\gamma'_o} \\ &= \gamma''\beta''\left(x'' + \frac{1}{\alpha''(w'')\gamma''^2_o}\right) - \frac{\beta''_o}{\alpha''_o\gamma''_o}, \\ x_I &= \gamma\left(x + \frac{1}{\alpha(w)\gamma_o^2}\right) - \frac{1}{\alpha_o\gamma_o} = \gamma'\left(x' + \frac{1}{\alpha'(w')\gamma'^2_o}\right) - \frac{1}{\alpha'_o\gamma'_o} \\ &= \gamma''\left(x'' + \frac{1}{\alpha''(w'')\gamma''^2_o}\right) - \frac{1}{\alpha''_o\gamma''_o}. \end{aligned} \tag{23}$$

The GT transformation between F and F' can be obtained from (23):

$$\begin{aligned}\beta(w) &= \frac{\gamma'\beta'Q' - \beta'_o/(\alpha'_o\gamma'_o) + \beta_o/(\alpha_o\gamma_o)}{\gamma'Q' - 1/(\alpha'_o\gamma'_o) + 1/(\alpha_o\gamma_o)},\\ x &= \sqrt{\left(\gamma'Q' - \frac{1}{\alpha'_o\gamma'_o} + \frac{1}{\alpha_o\gamma_o}\right)^2 - \left(\gamma'\beta'Q' - \frac{\beta'_o}{\alpha'_o\gamma'_o} + \frac{\beta_o}{\alpha_o\gamma_o}\right)^2} - \frac{1}{\alpha\gamma_o^2},\\ y &= y', \qquad z = z', \qquad Q' = \left(x' + \frac{1}{\alpha'(w')\gamma'^2_o}\right).\end{aligned} \tag{24}$$

Similarly, the GT transformation between F and F'' can be obtained from (23):

$$\begin{aligned}\beta(w) &= \frac{\gamma''\beta''Q'' - \beta''_o/(\alpha''_o\gamma''_o) + \beta_o/(\alpha_o\gamma_o)}{\gamma''Q'' - 1/(\alpha''_o\gamma''_o) + 1/(\alpha_o\gamma_o)},\\ x &= \sqrt{\left(\gamma''Q'' - \frac{1}{\alpha''_o\gamma''_o} + \frac{1}{\alpha_o\gamma_o}\right)^2 - \left(\gamma''\beta''Q'' - \frac{\beta''_o}{\alpha''_o\gamma''_o} + \frac{\beta_o}{\alpha_o\gamma_o}\right)^2} - \frac{1}{\alpha\gamma_o^2},\\ y &= y'', \qquad z = z'', \qquad Q'' = \left(x'' + \frac{1}{\alpha''(w'')\gamma''^2_o}\right).\end{aligned} \tag{25}$$

From (24) and (25), one can show that the GT transformation between $F'(x')$ and $F''(x'')$ is given by

$$\begin{aligned}\beta'(w') &= \frac{\gamma''\beta''Q'' - \beta''_o/(\alpha''_o\gamma''_o) + \beta'_o/(\alpha'_o\gamma'_o)}{\gamma''Q'' - 1/(\alpha''_o\gamma''_o) + 1/(\alpha'_o\gamma'_o)},\\ x' &= \sqrt{\left(\gamma''Q'' - \frac{1}{\alpha''_o\gamma''_o} + \frac{1}{\alpha'_o\gamma'_o}\right)^2 - \left(\gamma''\beta''Q'' - \frac{\beta''_o}{\alpha''_o\gamma''_o} + \frac{\beta'_o}{\alpha'_o\gamma'_o}\right)^2} - \frac{1}{\alpha'\gamma'^2_o},\\ y' &= y'', \qquad z' = z''.\end{aligned} \tag{26}$$

We have seen that the transformation (26) for two GLA frames, F' and F'', has the same form as that in (24) for F and F' frames, as required for the set of GT transformations to form a group. We can also show that the transformations $F_I \to F \to F' \to F''$ satisfies the associative rule.

The above results imply that the general taiji transformations form a group, which may be called ‘general taiji (GT) group’ with one arbitrary acceleration function $\alpha(w)$ and two parameters, i.e., the initial acceleration α_o and the initial velocity β_o. Mathematically, these group properties can also be shown by using the local GT transformation (13), which can be written in the matrix form.

The GT transformation (12) reduces to the Wu transformation (1) when the jerk vanishes, $J_e = 0$ or $\alpha_o = constant$. If the initial velocity also vanishes, $J_e = \beta_o = 0$, (12) further reduces to the Møller transformation, provided one makes a change of time variable [2]. Moreover, if $\alpha(w) = a_o \to 0$, the GT transformation (12) reduces to the 4-dimensional transformations (2). In this sense, the GT transformation for frames wth an arbitrary acceleration is the most general transformation for moton along a straight line.

For general motions in x, y and z directions, the group properties, the infinitesimal generators and the Lie algebra associated with the general taiji transformations are very complicated due to the presence of the jerk. Note that the totality of the differentials dx^μ at any point forms the cotangent space of differentials at the point, and that the coordinate frame $F(w, x, y, z)$ is the underlying spacetime. For simplicity, we may only consider transformations of the cotangent space at a fixed point with zero jerk. In this case, we have group properties and closed Lie algebra for general motions along x, y, and z axes for variable differentials at a fixed point.

V. Physical implications

Limiting 4-dimensional symmetry dictates that the coordinate system for writing basic covariant equations and transformations for GLA frames is the taiji spacetime coordinate system defined by the metric tensors in (15). This is analogous to the fact that pseudo-Cartesian coordinates are the preferred coordinates for the Lorentz transformations. In a GLA frame, time w is restricted by $\beta^2(w) < 1$ and space is limited by $x > -1/(\alpha(w)\gamma_o^2)$, as shown in the GT transformations (12) and (17).

We stress that GLA frames actually include both inertial and non-inertial frames with an arbitrary linear acceleration. There is no relativity between frames of reference in general, except in the limiting case in which the accelerations vanish. The GT transformations (12) provide foundations for the physics and the geometry of spacetime in all inertial frames and non-inertial frames with arbitrary accelerations along a straight line. Mathematically, the existence of these transformations implies that the Riemann curvature tensor for spacetime in all GLA frames vanish. In other words, the physical spacetime in both inertial and non-inertial frames are flat.

Note that the usual time t (measured in, say, seconds) and the universal constant c (measured in, say, cm/sec) do not exist within this theory in general. The physical time w, called taiji time, has the dimensional of length and can be realized physically by 'computerized clocks [5]. Since the variable speed of light is given by the function (19), it is very complicated to use such a non-constant speed of light to synchronize clocks in a GLA frame $F(w, x, y, z)$. However, it is not necessary in general to use light signals to synchronize clocks [2]. If the computer chips are not affected by acceleration or the effect due to acceleration can be corrected, then one can use a grid of 'computer clocks' in F to realize the taiji-time w: Namely, suppose a computer clock can accept information concerning its position in the F_I frame, obtain w_I from the nearest F_I clock, and then compute and display w using the inverse transformation of (12) [see also (17) and its explanations.

Let us consider classical electrodynamics in GLA Frames. Since the speed of light c in an accelerated frame F is no longer a universal constant, the invariant action for a charged particle moving in the electromagnetic 4-potential $a_\mu(x)$ is assumed to be [2]

$$S = \int (-mds - \overline{e} a_\mu dx^\mu) - \frac{1}{4}\int f_{\mu\nu} f^{\mu\nu} \sqrt{-det g_{\alpha\beta}} \;\; d^4x, \tag{27}$$

where the invariant charge $\overline{e}$ and other quantitier given by

$$\overline{e} = -1.6021891 \times 10^{-20}\sqrt{4\pi}\sqrt{g \cdot cm},$$

$$\sqrt{-det\ g_{\alpha\beta}} = \sqrt{W^2 + U^2} = \gamma^2\left(\alpha x + \tfrac{1}{\gamma_o^2}\right) > 0,$$

$$
\begin{aligned}
f_{\mu\nu} &= D_\mu a_\nu - D_\nu a_\mu = \partial_\mu a_\nu - \partial_\nu a_\mu, \quad dx^\mu = (dw, dx, dy, dz), \\
Da_\mu &= a_{\mu;\nu} dx^\nu = D_\nu a_\mu dx^\nu, \\
D_\nu a_\mu &= \partial_\nu a_\mu - \Gamma^\rho_{\mu\nu} a_\rho, \quad D_\nu a^\mu = \partial_\nu a^\mu + \Gamma^\mu_{\nu\rho} a^\rho, \\
\Gamma^\rho_{\mu\nu} &= \frac{1}{2} g^{\rho\alpha}(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}) = \Gamma^\rho_{\nu\mu},
\end{aligned}
$$

in a GLA frame F. Note that the invariant action S has the dimension of mass times length. In the limit $\alpha(w) = \alpha_o \to 0$, the F frame becomes an inertial frame and, hence, $\overline{e}$ and a_μ correspond to the charge e (in electrostatic units) and the usual electromagnetic potential $A_\mu(ct, x, y, z)$ by the relations $\overline{e} \leftrightarrow e/c$ and $a_\mu \leftrightarrow A_\mu/c$ respectively [5, 2].

The Lagrange equations of motion for a charged particle can be derived from the invariant action (27). We obtain

$$
m\frac{Du_\mu}{ds} = \overline{e} f_{\mu\nu} u^\nu, \tag{28}
$$

where the 4-velocities u^μ and u_μ are given by

$$
u^\nu = \frac{dx^\nu}{ds}, \qquad u_\mu = g_{\mu\nu} u^\nu
$$

For a continuous charge distribution in space, the invariant action for the electromagnetic fields and their interaction is assumed to be

$$
S_{em} = -\int \left[a_\mu j^\nu + \frac{1}{4} f_{\mu\nu} f^{\mu\nu} \right] \sqrt{-det g_{\alpha\beta}} \;\; d^4x. \tag{29}
$$

In GLA frames, this action leads to the following Maxwell's equations

$$
\begin{aligned}
g^{\mu\nu} D_\mu f_{\nu\alpha} &= j_\alpha, \qquad \partial_\lambda f_{\mu\nu} + \partial_\mu f_{\nu\lambda} + \partial_\nu f_{\lambda\mu} = 0, \\
D_\alpha f_{\mu\nu} &= \partial_\alpha f_{\mu\nu} - \Gamma^\rho_{\mu\alpha} f_{\rho\nu} - \Gamma^\rho_{\nu\alpha} f_{\mu\rho}.
\end{aligned} \tag{30}
$$

The canonical momentum P_i and the "Hamiltonian" $H = P_0$ of a particle in the GLA frame F are defined by

$$
\begin{aligned}
P_i &= -\frac{\partial L}{\partial(dx^i/dw)} = m g_{i\mu} \frac{dx^\mu}{ds} + \overline{e} a_i, \quad P_i = -P^i, \quad i = 1, 2, 3; \\
H &= \frac{\partial L}{\partial(dx^i/dw)} \frac{dx^i}{dw} - L = m g_{0\mu} \frac{dx^\mu}{ds} + \overline{e} a_0 \equiv P_0,
\end{aligned} \tag{31}
$$

where we have used the Lagrangian L for a charged particle given by the relation [8],

$$
(-m ds - \overline{e} a_\mu dx^\mu) = \left[-m\sqrt{g_{\mu\nu} dx^\mu dx^\nu} - \overline{e} a_0 - \overline{e} a_i \frac{dx^i}{dw} \right] dw = L dw. \tag{32}
$$

One can verify that the covariant 4-momentua P_i and P_0 satisfy the invariant relation

$$
g^{\mu\nu}(P_\mu - \overline{e} a_\mu)(P_\nu - \overline{e} a_\nu) = m^2. \tag{33}
$$

The contravariant momentum $p^\mu = mdx^\mu/ds = g^{\mu\nu}p_\nu$ transforms like the contravariant coordinate vector dx^μ in (13),

$$p_I^0 = \gamma(W_a p^0 + \beta p^1), \quad p_I^1 = \gamma(p^1 + \beta W_b p^0),$$
$$p_I^2 = p^2, \qquad p_I^3 = p^3. \tag{34}$$

For a 'non-interacting' particle at rest in $F(x)$ at the position $(x,0,0)$, its energy p_I^0 measured in F_I is $p_I^0 = \gamma W_a p^0 = m\gamma W_a/W$, where we have used $g_{\mu\nu}p^\mu p^\nu = m^2$, $p^i = 0, i = 1,2,3$, and (15). Thus, the change of energy per unit length is

$$\left(\frac{d}{dx_I}p_I^0\right)_{x\ fixed} = m\frac{\partial w}{\partial x_I}\frac{d}{dw}\left(\frac{\gamma W_a}{W}\right)_{x\ fixed}, \quad \frac{\partial w}{\partial x_I} = \frac{1}{\gamma\beta W_b}. \tag{35}$$

When the jerk is small, so that $W_a \approx W_b \approx W$, we have

$$\left(\frac{d}{dx_I}p_I^0\right)_{x\ \ fixed} \approx \frac{m\alpha}{(\alpha x + \gamma_o^{-2})}. \tag{36}$$

The invariant equation (33) for the momentum suggests that the generalized Klein-Gordon equation for GLA frames takes the form

$$g^{\mu\nu}(iJD_\mu - \overline{e}a_\mu)(iJD_\nu - \overline{e}a_\nu)\phi = m^2\phi, \quad iJD_\mu = P_\mu. \tag{37}$$

The limiting 4-dimensional symmetry dictates that the connection between between inertial and non-inertial frames must be smooth. Furthermore, the proportionality constant between the covariant 4-momentum P_μ (which has the dimension of mass) and the partial covariant derivative iD_μ must be the universal constant $J = 3.5177293 \times 10^{-38} g \cdot cm$. This is due to the fact that, in an inertial frame, P_μ has a correspondence with the usual momentum $P_\mu(usu)$, $P_\mu \leftrightarrow P_\mu(usu)/c$ and, hence, one has $J \leftrightarrow \hbar/c$ [2].

The Wu-Lee kinematic approach [4] based on limiting 4-dimensional symmetry [2] provides a new direction for the investigation of physics in general non-inertial frames with a simple mathematical structure of spacetime, namely, the Riemann curvature tensor vanishes (i.e., the spacetime is flat). Based on previous discussions, it is gratifying that the truly universal and fundamental constants in both inertial and non-inertial frames are the quantum constant, $J = 3.5177293 \times 10^{-38} g \cdot cm$, and the electric charge in the electromagnetic units, $\overline{e} = -1.6021891 \times 10^{-20}\sqrt{4\pi}\sqrt{g \cdot cm}$, [or $\alpha_e = \overline{e}^2/(4\pi J)$ = 1/137.036]. It is interesting that these universal constants for non-inertial frames with arbitrary-linear-accelerations turn out to be precisely the same as those in the theory of relativity which is formulated solely on the basis of the first principle of relativity, without making any assumptions concerning the speed of light [5]. It is hoped that these results can be experimentally tested with linear accelerators in the future.

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References

[1] For early discussions of acceleration transformations, see A. Einstein, Jahrb. Rad. Elektr. **4**, 411 (1907). [In his paper, Einstein asked: "Is it conceivable that the principle of relativity also holds for systems which are accelerated relative to each other?" He said that this question must occur to everyone who has followed the applications of the relativity principle. We demonstrate in the present paper that this question can be answered affirmatively in terms of the limiting 4-dimensional symmetry.] L. Page, Phys. Rev. **49**, 254 (1936); C. Møller, Danske Vid. Sel. Mat.-Fys. 20, No. 19 (1943); T. Fulton, F. Rohrlich and L. Witten, Nuovo Ciment, XXVI, 652 (1962) and Rev. Mod. Phys. **34**, 442 (1962); E. A. Desloge and R. J. Philpott, Am. J. Phys. **55**, 252 (1987).

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[5] Such a taiji time w appears naturally in the theory of spacetime for inertial frames based solely on the first postulate of relativity, without making any postulate regarding the speed of light. See Jong-Ping Hsu and Leonardo Hsu, Phys. Lett. **A196**, 1 (1994) and Ref. [2].

[6] The generalization with this replacement $\alpha_o \to \alpha(w)$ is crucial. Mathematically, this replacement implies that the two variables w and x in $W(w,x)$ cannot be separated. In contrast, a general transformation for GLA frames was discussed in a previous paper based on the separation of w and x in $W(w,x)$. See J. P. Hsu, in *FRONTIERS OF PHYSICS AT THE MILLENNIUM, SYMP* (Ed. Y. L. Wu and J. P. Hsu, World Scientific, Singapore). The generality in this conference paper turned out to be restricted and, hence, not completely satisfactory because the additional assumption of separation of variables, $W(w,x) = W_1(w)W_2(x)$, prevents the truly general linear accelerations from being full realized.

[7] See, for example, L. Landau and E. lifshitz, *The Classical Theory of Fields* (Addison-Wesley, 1951) p. 258.

[8] It is worthwhile noting that, since Ldw in (32) is invariant and dx^μ is a contravariant 4-vector, the Hamiltonian H defined in (31) turns out to be the zeroth component P_0 of the covariant 4-momentum vector P_μ. Furthermore, since the Hamiltonian is defined in (31), Ldw may be interpreted as the invariant infinitesimal phase, $Hdx^0 + P_i dx^i = Ldw$ (where $dw = dx^0$), associated with the quantum-mechanical wave of a particle.

(A) (B) (C) (D)

(A) (B) (C) (D)