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Squeezed Displaced Number States Defined in the Coherent-State Quantum Phase-Space Representation of Quantum Mechanics

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We derive the most general squeezed state in the Coherent-State quantum phase-space Representation of nonrelativistic quantum mechanics (CSR). The wave functions, time-dependent uncertainties, mean photon number, photon number variance, probability flux vectors, and probability densities are obtained.

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In recent years many authors have investigated new quantum states of the electromagnetic field; the squeezed state can be considered as one of the most studied states [1]. It is characterized by the fact that the quantum fluctuations in one of the two canonical components are smaller than those in the usual coherent states. Plebanski, in wave-function form, also looked into what we would call the squeezed number state [2]. The Wigner function is widely used for the calculation of expectation values [3-5] in phase-space; experimental reconstructions reveal a negative Wigner function in position-momentum space [6]. We will study in quantum phase-space a new class of squeezed states called Squeezed Displaced Number States (SDNS), recently introduced in the coordinate representation [7-8].

The CSR [9-10] allows us to analyze completely the dynamics of the quantum systems in the phase-space in the same way as in the coordinate representation. In this representation a Schrödinger like equation is employed together with the wave function $\psi(q,p) = \langle q,p \mid \psi\rangle$ defined within the real "phase-space" by simultaneous coordinates q and p. The quantity $\mid \psi(\Gamma) \mid^2 \equiv \psi^*(\Gamma)\psi(\Gamma)$ represents a probability density, where $\Gamma = (q,p)$ is a phase-space point. This definition ensures that the quantum density $\mid \psi(\Gamma) \mid^2$ is a nonnegative quantity in phase-space and fulfills all the requirements of a probability density. The operators associated to momentum $\hat{P}$ and coordinate $\hat{Q}$ are given by $\hat{P} = (p/2 - i\hbar\partial/\partial q)$, $\hat{Q} = (q/2 + i\hbar\partial/\partial p)$. These operators obey $[\hat{Q},\hat{P}] = i\hbar$. The calculation of the expectation value of the operator $\hat{A}$, defined in the CSR is carried out as in the usual definition, i.e., $\langle\hat{A}\rangle = \int d\Gamma\psi^*(\Gamma)\hat{A}\psi(\Gamma)$, where the integration is carried out the whole phase-space with $d\Gamma = dqdp$.

The diagonal matrix element of the quantum probability conservation equation (abstract quantum Liouville equation) in the CSR, allows us to calculate the phase-space components of the probability flux vector. When we employ the harmonic oscillator potential, they are given by

$$\begin{aligned} J_q &= \frac{1}{2}\left[\langle\Gamma \mid \hat{P} \mid \psi\rangle\langle\psi \mid \Gamma\rangle + \langle\Gamma \mid \psi\rangle\langle\psi \mid \hat{P} \mid \Gamma\rangle\right], \\ J_p &= -\frac{1}{2}\left[\langle\Gamma \mid \hat{Q} \mid \psi\rangle\langle\psi \mid \Gamma\rangle + \langle\Gamma \mid \psi\rangle\langle\psi \mid \hat{Q} \mid \Gamma\rangle\right]. \end{aligned} \tag{1}$$

The SDNS, $\mid n, \beta, \xi\rangle$ is obtained by the application of the general squeeze operator $\hat{S}(\xi)$ followed by the displacement operator $\hat{D}(\beta)$, on the number state $\mid n\rangle$, i.e., $\mid n, \beta, \xi\rangle = \hat{D}(\beta)\hat{S}(\xi) \mid n\rangle$, where

$$\hat{S}(\xi) = \exp(\xi\hat{a}^{\dagger 2} - \xi^{*}\hat{a}^{2}), \quad \text{and} \quad \hat{D}(\beta) = \exp(\beta\hat{a}^{\dagger} - \beta^{*}\hat{a}), \tag{2}$$

in accordance with Nieto [8]. Here β is the coherence parameter, $\beta = (q_o + ip_o)/\sqrt{2} = \beta_o \exp(i\theta_o)$, and ξ is the squeezing parameter, $\xi = \eta\exp(i\theta)$. The annihilation and creation operators in the CSR, can be written with the help of the previously defined $\hat{Q}$ and $\hat{P}$. For simplicity in the following, we will use $\hbar = \omega = m = 1$.

Using the Quantum Commutation Relations of the generators of Canonical Transformations (QCRCT) [11] (or equations (35) in reference [4]), the Baker-Campbell-Hausdorff relation (BCH) [12] and the functional forms (2), of $\hat{D}$ and $\hat{S}$, we find that the most general SDNS in quantum phase-space are

$$\begin{aligned} \langle\Gamma, q_o, p_o, \xi \mid n\rangle = &\mathcal{N}\, H_n(\Gamma, q_o, p_o; \alpha)\exp\Big\{-\frac{1}{2}\phi(q-q_o)^2 \\ &-\frac{1}{2}\gamma(p-p_o)^2 + i[\gamma p_o q - \phi p q_o - \alpha p q + \phi p_o q_o]\Big\}, \end{aligned} \tag{3}$$

where the number states $\mid n\rangle$ are defined in [10], $\mathcal{N}$ is a normalization constant, $\gamma = \frac{1}{2} + \alpha$, $\phi = \frac{1}{2} - \alpha$, and α is a complex-valued parameter defined by $\alpha = -\frac{1}{2}\tanh(2\eta)\exp(-i\theta) = \alpha_o\exp(-i\theta)$, with $\mid \alpha \mid < \frac{1}{2}$. $H_n(\Gamma, q_o, p_o; \alpha)$, are a set of orthogonal polynomials in phase space with similar properties, and which can be reduced to the usual Hermite polynomials. They satisfy the recursion relationship

$$H_{n+1}(\Gamma, q_o, p_o; \alpha) = 2u(\Gamma, q_o, p_o; \alpha)H_n(\Gamma, q_o, p_o; \alpha) - 4n\alpha^{*}H_{n-1}(\Gamma, q_o, p_o; \alpha),$$

where, $u(\Gamma, q_o, p_o; \alpha) = (\frac{1}{4} - \mid \alpha \mid^2)^{1/2}[(q-q_o) - i(p-p_o)]$. Some of these polynomials are:

$$\begin{aligned} H_0(\Gamma, q_o, p_o; \alpha) &= 1, \\ H_1(\Gamma, q_o, p_o; \alpha) &= 2u(\Gamma, q_o, p_o; \alpha), \\ H_2(\Gamma, q_o, p_o; \alpha) &= 4u^2(\Gamma, q_o, p_o; \alpha) - 4\alpha^{*}, \\ H_3(\Gamma, q_o, p_o; \alpha) &= 8u^3(\Gamma, q_o, p_o; \alpha) - 24u(\Gamma, q_o, p_o; \alpha)\alpha^{*}. \end{aligned}$$

The time-dependent uncertainties can be calculated by means of the propagator $\exp(-i\hat{H}t) =$

$\exp[-it(\hat{Q}^2+\hat{P}^2)/2]$, the QCRCT, the BCH relation and the wave function (3). Then, specifically

$$\begin{aligned}\left(\frac{1}{4}-\alpha_o^2\right)\langle(\Delta\hat{Q})^2\rangle &= \left(\frac{1}{2}+n\right)\left[\left(\frac{1}{2}+\alpha_o\cos(\theta-2t)\right)^2+\alpha_o^2\sin^2(\theta-2t)\right],\\ \left(\frac{1}{4}-\alpha_o^2\right)\langle(\Delta\hat{P})^2\rangle &= \left(\frac{1}{2}+n\right)\left[\left(\frac{1}{2}-\alpha_o\cos(\theta-2t)\right)^2+\alpha_o^2\sin^2(\theta-2t)\right],\end{aligned} \tag{4}$$

therefore, the uncertainty-product as a function of time is

$$\langle(\Delta\hat{Q})^2\rangle\langle(\Delta\hat{P})^2\rangle = \left(\frac{1}{2}+n\right)^2[1+\sinh^2 4\eta\sin^2(\theta-2t)]. \tag{5}$$

The photon number $\hat{N}$ and the squared photon number $\hat{N}^2$ are the most important operators in quantum optics. According to the harmonic oscillator Hamiltonian $\hat{N}=\hat{H}-1/2$, the QCRCT, the BCH relation and the wave function (3), we can write

$$\langle\hat{N}\rangle = n+\frac{1}{2}(p_o^2+q_o^2)+(2n+1)\langle\hat{N}\rangle_o, \tag{6}$$

$$\begin{aligned}\langle(\Delta\hat{N})^2\rangle =&(n^2+n+1)\langle(\Delta\hat{N})^2\rangle_o\\ &+\frac{1}{2}(p_o^2+q_o^2)(2n+1)[\cosh 4\eta-\sinh 4\eta\cos(\theta-2\theta_o)],\end{aligned} \tag{7}$$

where

$$\langle\hat{N}\rangle_o = \frac{\alpha_o^2}{\frac{1}{4}-\alpha_o^2} = \frac{1}{2}(\cosh 4\eta-1), < (\Delta\hat{N})^2\rangle_o = \frac{1}{2}\frac{\alpha_o^2}{\left(\frac{1}{4}-\alpha_o^2\right)^2} = \frac{1}{2}\sinh^2 4\eta.$$

For $t=0$, equations (4-7) agree with the standard squeezed-state result when $n=0$ [5], in the Wigner representation and the phase-space squeezed state defined in [11]. Equations (6, 7) agree with the results defined in [7].

In figure 1 we show, by means of density plots, the shape in phase-space of the square magnitude of the SDNS, initially centered at $(q_o,p_o)=(1,1)$, (see figure 1(a)), with $n=3$ and $\alpha=0.25\exp(i\pi/2)$, in which darker regions indicate larger values of the density. In this density plot the height increases as a power of 2.0 in order to show the parts with a small density value. In this way we can visualize the regions where the density is as small as 0.018 of the maximum height. We can see that the probability density from the SDNS is rotated by an angle $\theta/2$ in a counter clockwise sense, where θ is the argument of α. In figure 1(b), we show the quantum evolution in phase-space of the probability flux vector corresponding to the SDNS shown in figure 1(a), calculated by means of the equation (1) at time $t=0$, (here $\langle\Gamma\mid\psi\rangle$ is the equation (3), we draw for simplicity only the density contours), the size of the vectors are proportional to their magnitudes. These vectors assemble a non-stationary vortex, allowing the SDNS density to rotate in a clockwise sense around the "origin" in phase space, i.e., for time $t=\pi/2$ (see figure 1(c)), the square magnitude of the SDNS is rotated by angle $\pi/2$ in a clockwise sense, also shown with

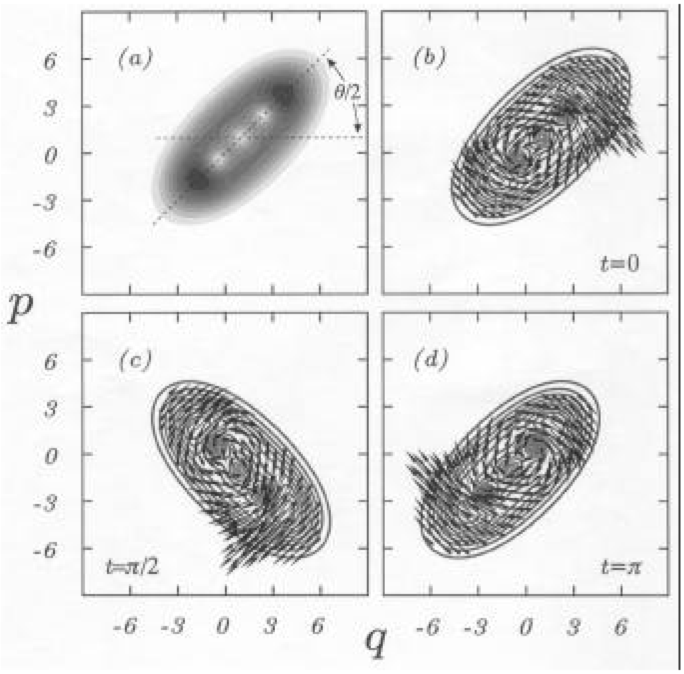

FIG. 1. Snapshots of the quantum evolution in phase space of a squeezed displaced number state (SDNS) and its probability flux vector moving in a harmonic potential. The initial density was centered at $(q_o, p_o) = (1, 1)$, with $n = 3$ and $\alpha = 0.25\exp(i\pi/2)$.

the help of the time evolution operator $\exp(-it\hat{H})$. For $t = \pi$ (figure (1(d)) the corresponding square magnitudes rotate by an angle of π in a clockwise sense due to the non-stationary vortex.

The photon number distribution (see. e.g., [7]), corresponding to the SDNS and representing the probability of finding n photons in the SDNS is easily obtained employing equation (3) and the recursion relationship. In principle, we can control the quantum phase-space interference [13] and the photon number distribution.

In conclusion, we find this representation particularly useful for our purpose since, on one hand, it allows for the analysis of quantum dynamics in phase space in terms of wave functions, provides a way to make quantitative analysis as in the Schrödinger representation and simplifies considerably the calculations, in comparison with others in the literature [5]; on the other hand the square magnitude of the phase-space wave function is analogous to the Husimi density [14] which then is a tool for comparing classical and quantum dynamics. Although the states SNDS have not been obtained experimentally, there are techniques for generating coherent (displaced) states [15]. They also have been able to produce number states and squeezed (but not displaced) ground states [6]. If the above techniques can be combined, then, in the not too distant future SDNS can be observed and we hope they should mimic in the phase-space the features in our figure 1.

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