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Singularity Structure Analysis and Abundant New Dromion-like Structures for the (2+1)-Dimensional Generalized Burgers Equation

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The Painleve (P) singularity analysis method (the WTC method, due to Weiss, Tabor and Carnevale) is a powerful tool for proving the P-property of nonlinear partial differential equations and their Bäcklund transformations. In this paper, the singularity structure analysis is performed for the (2+1)-dimensional generalized Burgers equation, $u_t + u_{xy} + uu_y + u_x\partial_x^{-1}u_y = 0$, by using the WTC method; it is shown that the equation passes the Panilevé test. Based on the P-analysis, a Bäcklund transformation is obtained, and then it is used to find many exact solutions including N-soliton-like solutions and new exact solutions. Some of these obtained solutions are used to prove that the variable $u_y(x, y, t)$, rather than the physical field $u(x,y,t)$ in the (2+1)-dimensional generalized Burgers equation, admits abundant dromion-like solutions (exponentially localized solutions) such as point dromions, ring dromions, extended dromions, sharp dromions and oscillatory dromion solutions.

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I. Introduction

Finding new nonlinear evolution equations (NLEEs) and their abundant exact solutions has been an important subject in soliton theory and its applications since the solitary wave phenomena observed by Scott Russell [1] in 1834. Early in the study of soliton theory, the main interests of peoples were to pay attention to the (1+1)-dimensional cases, such as the KdV equation, the Burgers equation, the Boussinesq equation, etc. [1-3]. Since the concept of dromions (the exponentially localized solutions in (2+1)-dimensional space) was introduced by Boiti *et al.* [4], the study of soliton-like solutions in higher dimensions has attracted much more attention. Many nonlinear soliton equations in (2+1)-dimensional space, such as the generalized KdV equation, the Kadomtsev-Petviashvilli equation, the Davey-Stewartson (DS) equation, the dispersive long wave equation, and the Nizhnik-Novikov-Veselov equation possess dromions [4-12]. Recently, Lou [13] constructed a (3+1)-dimensional KdV-type equation

$$w_t + 6w_xw_y + w_{xxy} + w_{xxxxz} + 60w_x^2w_z + 10w_zw_{xxx} + 20w_xw_{xxz} = 0. \tag{1}$$

Though it was shown that Eq. (1) does not possess the Painleve property, Lou found abundant dromion-like structures which contain multi-soliton solutions. More recently Lou [14, 15] studied abundant structures of the Nizhnik-Novikov-Veselov equation by using the separation of variables approach.

In this paper, we would like to consider the following (2+1)-dimensional generalized Burgers equation:

$$u_t + u_{xy} + uu_y + u_x\partial_x^{-1}u_y = 0. \tag{2}$$

When $y = x$, Eq. (2) reduces to be well-known Burgers equation. To our knowledge, Painlevé integrability and exact dromion-like solutions of this equation were not studied. A natural question is whether the equation (2) also possesses dromions or dromion-like solutions. In order to answer the question, we firstly perform the singularity analysis structure of Eq. (2) such that a Bäcklund transformation (BT) is obtained. N-soliton-like solutions and infinitely many rational solutions of Eq. (2) are derived based on the obtained BT. From the solutions obtained, it is easy to show that the variable u_x rather than its physical field $u(x, y, t)$ admits exponentially localized solutions.

The rest of this paper is organized as follows: In Sec. II. we perform a Painleve analysis of the (2+1)-dimensional generalized Burgers equation by using the WTC method [16] such that it is proved that the equation possesses Painleve property. In Sec. III, based on the result in Sec. II, a Bäcklund transformation (BT), the Cole-Hopf transformation and a non-local symmetry of Eq. (2) are obtained. In Sec. IV, we give some N-soliton-like solutions and other new solutions of Eq. (5) based upon the obtained BT. In addition, we analyze the properties of some obtained solutions. It is shown that the variable u_y rather than its physical field $u(x, y, t)$ admits exponentially localized solutions [5]. Finally, some conclusions are given in Sec. V.

II. Painlevé singularity structure analysis

In order to investigate the singularity structure aspects of Eq. (2) [16], we shall introduce the transformation $u_x = w_y$ and change Eq. (2) into a set of two coupled nonlinear partial differential equations

$$u_t + u_{xy} + uw_x + u_xw = 0, \tag{3a}$$

$$u_x = w_y. \tag{3b}$$

In the coupled equation, $u(x, y, t)$ denotes the physical field and $w(x, y, t)$ some potential. in order to carry out a singularity structure analysis, we effect a local Laurent expansion in the neighborhood of a non-characteristic singular manifold, $\phi(x, y, t) = 0, (\phi_x, \phi_y \neq 0)$. Assuming the leading orders of the solutions of system (3) to have the form

$$u = u_0\phi^\alpha, \quad w = w_0\phi^\beta, \tag{4}$$

where u_0 and w_0 are analytic functions of x, y, t and α and β are integers to be determined later, substituting the above equation into system (3) and balancing the nonlinear terms against the dominant linear terms, we have

$$\alpha = \beta = -1, \quad u_0 = \phi_x, \quad w_0 = \phi_y. \tag{5}$$

Now, considering the Laurent series expansion of the solutions in the neighborhood of the singular manifold [13]

$$u = \sum_{j=0}^{\infty} u_j\phi^{j-1}, \quad w = \sum_{j=0}^{\infty} w_j\phi^{j-1}. \tag{6}$$

Substituting Eq. (6) into system (3), yields a recursion relation for u_j and w_j

$$\begin{aligned}&(j-2)u_{j-1}\phi_t + u_{j-2,t} + (j-1)(j-2)u_j\phi_x\phi_y + (j-2)u_{j-1}\phi_{xy} + (j-2)u_{j-1,y}\phi_x \\ &\quad +(j-2)u_{j-1,x}\phi_y + u_{j-2,xy} + \sum_{i=0}^{j} u_{j-i}[w_{i-1,x} + (i-1)w_i\phi_x] \\ &\quad + \sum_{i=0}^{j} w_{j-i}[u_{i-1,x} + (i-1)u_i\phi_x] = 0,\end{aligned} \tag{7a}$$

$$(j-1)u_j\phi_y + u_{j-1,y} = (j-1)w_j\phi_x + w_{j-1,x}. \tag{7b}$$

The resonances, that is, the powers at which an arbitrary function occurs can be obtained by comparing the coefficients of (ϕ^{j-3}, ϕ^{j-2}), that is to say

$$\begin{pmatrix} j(j-2)\phi_x\phi_y & (j-2)\phi_x^2 \\ (j-1)\phi_y & -(j-1)\phi_x \end{pmatrix} \begin{pmatrix} u_j \\ w_j \end{pmatrix} = 0. \tag{8}$$

Evaluating Eq. (8), we arrive at the resonances

$$j = -1, 1, 2. \tag{9}$$

The resonance at $j = -1$ represents the arbitrariness of the singularity manifold $\phi(x, y, t) = 0$. In what follows, let us prove the existence of arbitrary functions in the other two cases $j = 1, 2$ successively.

Case (i): *The case* $j = 1$ (*resonance*)
The system (7) for $j = 1$ reduces to

$$\phi_x\phi_y u_1 + \phi_x^2 w_1 = -\phi_x\phi_t - \phi_x\phi_{xy}, \tag{10}$$

$$u_{0y} = w_{0x}. \tag{11}$$

By the use of Eq. (5), we can easily verify that Eq. (11) identically vanishes and hence we are left with only one equation (10) for two variables u_1 and w_1 and so one of them must be arbitrary.

Case (ii): *The case* $j = 2$ (*resonance*)
Proceeding further to consider system (7), the system (7) for $j = 2$ becomes

$$u_{0t} + u_{0xy} + u_1w_{0x} + u_0w_{1x} + w_1u_{0x} + w_0u_{1x} = 0, \tag{12}$$

$$\phi_y u_2 - \phi_x w_2 = w_{1x} - u_{1y}. \tag{13}$$

It is easy to see that Eq. (11a) is independent of u_2 and w_2. From Eqs. (10) and (12), we know that there exist u_1 and w_1 such that Eqs. (10) and (12) hold. Hence we are left with only one equation (13) for two variables u_2 and w_2 and so one of them must be arbitrary. Therefore the general solution $(u(x, y, t), w(x, y, t))$ of (3) admits the required number of arbitrary functions, without the introduction of any movable critical manifold, thereby satisfying the Painlevé property. Thus the equation (2) passes the Painlevé test.

III. Leading to a Bäcklund transformation

In order to construct a Bäcklund transformation for Eq. (2), we truncate the Laurent series at the constant level term, that is, $u_j = w_j = 0$, for $j \geq 2$, to give rise to

$$u(x, y, t) = u_0\phi^{-1} + u_1 = \partial_x \ln\phi + u_1, \tag{14a}$$

$$w(x, y, t) = w_0\phi^{-1} + w_1 = \partial_y \ln\phi + w_1. \tag{14b}$$

Substituting Eq. (14b) into Eq. (3b) yields

$$u_{1y} = w_{1x}, \tag{15}$$

with which Eq. (14b) becomes

$$w(x, y, t) = w_0\phi^{-1} + w_1 = \partial_y \ln\phi + \partial_x^{-1}u_{1y} = \partial_x^{-1}u_y(x, y, t). \tag{16}$$

Substituting Eqs. (14a) and (16) into Eq. (3a), yields

$$\begin{aligned}
&-\phi_x(\phi_t + \phi_{xy} + u_1\phi_y + \phi_x\partial^{-1}u_{1y})\phi^{-2}\\
&+(\phi_{xt} + \phi_{xxy} + u_{1x}\phi_y + u_{1y}\phi_x + u_1\phi_{xy} + \phi_{xx}\partial_x^{-1}u_{1y})\phi^{-1}\\
&+u_{1t} + u_{1xy} + u_1u_{1y} + u_{1x}\partial_x^{-1}u_{1y} = 0.
\end{aligned} \tag{17}$$

According to the linearly independence of ϕ^{-2}, ϕ^{-1} and $\phi^0 = 1$, we have the following system of over-determined partial differential equations (SOPDEs) w.r.t. $\phi(x, y, t)$ and $u_1(x, y, t)$.

$$-\phi_x(\phi_t + \phi_{xy} + u_1\phi_y + \phi_x\partial^{-1}u_{1y}) = 0, \tag{18.1}$$

$$\phi_{xt} + \phi_{xxy} + u_{1x}\phi_y + u_{1y}\phi_x + u_1\phi_{xy} + \phi_{xx}\partial_x^{-1}u_{1y} = 0, \tag{18.2}$$

$$u_{1t} + u_{1xy} + u_1u_{1y} + u_{1x}\partial_x^{-1}u_{1y} = 0. \tag{18.3}$$

Because it is clear to see that u_1 is just a solution of Eq. (2) from Eq. (18.3), Eq. (14a) is just an auto-Bäcklund transformation (BT) [2] for Eq. (2) with ϕ and u_1 satisfying Eq. (18).

Remark i: If we set $u_1 = 0$ in BT (14a), then BT (14a) becomes the famous Cole-Hopf transformation [2] of Eq. (2), $u(x, y, t) = \frac{\phi_x}{\phi}$, under which a bilinear equation of Eq. (2) is given by

$$\phi\phi_{xt} - \phi_x\phi_t + \phi\phi_{xxy} - \phi_x\phi_{xy} = 0. \tag{19}$$

Remark ii: In order to use Eq. (2), we rewrite Eq. (2) to be

$$u_t = K[u] = -u_{xy} - uu_y - u_x\partial_x^{-1}u_y. \tag{20}$$

It is well known that σ is called a symmetry of Eq. (2), if σ satifies the following linear equation

$$\sigma_t = K'[u]\sigma = -[\partial_{xy} + u_y + u\partial_y + u_x\partial_x^{-1}\partial_y + (\partial_x^{-1}u_y)\partial_x]\sigma, \tag{21}$$

where u satisfies Eq. (2), $\partial_x = \partial/\partial x, \partial_y = \partial/\partial y, \partial_x^{-1} = \int dx, \partial_x^{-1}\partial_x = \partial_x\partial_x^{-1} = 1$ and $K'[u]$ denotes the Frechet derivative of $K[u]$, that is

$$K'[u] = \frac{d}{d\mu}K[u + \mu\sigma]|_{\mu=0}.$$

From Eq. (18.2), it is easy to show that

$$\begin{aligned}(\phi_x)_t &= -[\partial_{xy} + u_y + u\partial_y + u_x\partial_x^{-1}\partial_y + (\partial_x^{-1}u_y)\partial_x]|_{u=u_1}\phi_x \\ &= K'[u]|_{u=u_1}\phi_x = K'[u_1]\phi_x.\end{aligned} \tag{22}$$

Because u_1 satisfies Eq. (2), therefore we know that $\sigma = \phi_x$ is a non-local symmetry of Eq. (2).

IV. Abundant exact solutions and their analysis

By the virtue of the BT (14a), we reduce Eq. (2) to a system of over-determined partial differential equations (18.1)-(18.3). If we can obtain the solution (u_1, ϕ) of (18.1)-(18.3), then a new solution u of Eq. (2) is obtained by using the BT (14a). We now mainly consider SOPDEs (18).

Case 1: When $u_1(x,t) = u_1(x)$. Thus Eqs. (18.1)-(18.3) reduce to

$$\phi_t + \phi_{xy} + u_1(x)\phi_y = 0, \tag{23}$$

which has the solution

$$\phi(x,y,t) = a_0 + a_1 \exp \square\ \lambda x + \int^x u_0(s)ds - \theta y - \lambda\theta t\Big], \tag{24}$$

where $a_i (i = 0, 1), \theta$ and λ are arbitrary real constants and $u_1(x)$ is an integrable function of x.

Substituting Eq. (24) into Eq. (14a), the generalized soliton-like solutions of Eq.(2) are

$$u(x,y,t) = \frac{a_1(\lambda + u_0(x))\exp[\lambda x + \int^x u_0(s)ds - \theta y - \lambda\theta t]}{a_0 + a_1\exp[\lambda x + \int^x u_0(s)ds - \theta y - \lambda\theta t]} + u_1(x); \tag{25}$$

(25) becomes the shock-like wave solution

$$\begin{aligned}u_1 = &\frac{\lambda + u_1(x)}{2}\tanh\frac{1}{2}\ \square\ \lambda x + \int^x u_0(s)ds - \theta y - \lambda\theta t - \ln\frac{a_1}{a_0}\Big] \\ &+\frac{1}{2}\lambda + \frac{3}{2}u_1(x), \quad a_0a_1 > 0\end{aligned} \tag{26}$$

and singular soliton-like solution

$$\begin{aligned} u_2 =& \frac{\lambda + u_1(x)}{2} \coth \frac{1}{2} \left[\lambda x + \int^x u_0(s) ds - \theta y - \lambda \theta t - \ln \left| \frac{a_1}{a_0} \right| \right] \\ &+ \frac{1}{2}\lambda + \frac{3}{2} u_1(x), \quad a_0 a_1 < 0. \end{aligned} \tag{27}$$

Obviously, when $u_1(x) = const.$, solution (26) becomes a kink soliton solution. Since $u_1(x)$ is an arbitrary function of x, the solution (26) possesses rich structures. Here are three simple examples:

Type 1: When we take $u_1(x) = \lambda sech^m(x - x_0)$, solution (26) is

$$\begin{aligned} u_1(x, y, t) =& \frac{\lambda + \lambda_1 sech^m(x - x_0)}{2} \tanh \frac{1}{2} \\ & \left[\lambda x + \int^x \lambda sech^m(s - x_0) ds - \theta y - \lambda \theta t - \ln \frac{a_1}{a_0} \right] \\ & + \frac{1}{2}\lambda + \frac{3}{2}\lambda sech^m(x - x_0), \quad a_0 a_1 > 0. \end{aligned} \tag{28}$$

Type 2: When we take $u_1(x) = \lambda sech^m[\cosh(x - x_0)]$, solution (26) is

$$\begin{aligned} u_2(x, y, t) =& \frac{\lambda + \lambda sech^m[\cosh(x - x_0)]}{2} \tanh \frac{1}{2} \\ & \left[\lambda x + \lambda \int^x sech^m[\cosh(s - x_0)] ds - \theta y - \lambda \theta t - \ln \frac{a_1}{a_0} \right] \\ & + \frac{1}{2}\lambda + \frac{3}{2}\lambda sech^m[\cosh(x - x_0)], \quad a_0 a_1 > 0. \end{aligned} \tag{29}$$

Type 3: When we take $u_1(x) = \frac{\lambda}{(x - x_0)^{2m} + 1}$, solution (26) is

$$\begin{aligned} u_2(x, y, t) =& \frac{1}{2} \left(\lambda + \frac{\lambda}{(x - x_0)^{2m} + 1} \right) \tanh \frac{1}{2} \\ & \left[\lambda x + \int^x \frac{\lambda}{(s - x_0)^{2m} + 1} ds - \theta y - \lambda \theta t - \ln \frac{a_1}{a_0} \right] \\ & + \frac{1}{2}\lambda + \frac{3}{2} \frac{\lambda}{(x - x_0)^{2m} + 1}, \quad a_0 a_1 > 0. \end{aligned} \tag{30}$$

The first type of solution, $u_1(x, y, t)$, decays exponentially in the x direction. The second type of solution, $u_2(x, y, t)$, decays much more quickly than the first in the x direction. The third type of solution, $u_3(x, y, t)$, decays much slower than the first in the x direction. But it is obvious to see that, when $\theta \to 0$, the three types of solutions survive as three functions which are not identical to zero. Therefore, the variable u does not admit exponentially localized solutions, that is to say, the solution $u_i (i = 1, 2, 3)$ are not dromion solutions.

But we have from Eqs. (28)-(30)

$$\begin{aligned} u_{1y}(x,y,t) =& -\frac{\theta[\lambda+\lambda sech^m(x-x_0)]}{2} sech^2\frac{1}{2} \\ & \square \\ & \lambda x+\int^x \lambda sech^m(s-x_0)ds-\theta y-\lambda\theta t-\ln\frac{a_1}{a_0}\Bigg], \end{aligned} \tag{31}$$

$$\begin{aligned} u_{2y}(x,y,t) =& -\frac{\theta[\lambda+\lambda sech^m[\cosh(x-x_0)]]}{2} sech^2\frac{1}{2} \\ & \square \\ & \lambda x+\lambda\int^x sech^m[\cosh(s-x_0)]ds-\theta y-\lambda\theta t-\ln\frac{a_1}{a_0}\Bigg], \end{aligned} \tag{32}$$

$$\begin{aligned} u_{3y}(x,y,t) =& -\frac{1}{2}\theta\left(\lambda+\frac{\lambda}{(x-x_0)^{2m}+1}\right) sech^2\frac{1}{2} \\ & \square \\ & \lambda x+\int^x \frac{\lambda}{(s-x_0)^{2m}+1}ds-\theta y-\lambda\theta t-\ln\frac{a_1}{a_0}\Bigg]. \end{aligned} \tag{33}$$

The first type of solution, u_{1y}, decays exponentially in all directions. The second type of solution, u_{2y}, decays much more quickly than the first in the x direction. The third type of solution, u_{3y}, decays much slower than the first in the y direction. If $u_0(x)$ is selected as N parallel line solitons (parallel to y-axis), then we have an N-dromion bound state

$$u_y=-\frac{\theta_1[\lambda+\lambda_1\sum_{j=1}^{N} f_j(x)]}{2} sech^2\frac{1}{2}\left[\lambda x+\lambda\int^x \sum_{j=1}^{N} f_j(x)-\theta y-\lambda\theta t-\ln\frac{a_1}{a_0}\right], \tag{34}$$

derived from N line ghost solitons and one curved line ghost soliton.

Case 2: When $u_1 = const.$ we can obtain a solution of system (18)

$$\phi(x,y,t)=\mu+\sum_{j=1}^{N}\exp[k_j x+g_i(\xi_{i1},...,\xi_{in})], \quad \mu=\pm 1, \tag{35}$$

where the g_i are arbitrary functions of the variables

$$\xi_{ij}=\lambda_{ij}y-\lambda_{ij}(k_j+u_1)t, \quad (i=1,....,N; j=1,...,n),$$

and k_j, u_1, λ_{ij} are constants.

Substituting Eq. (35) into Eq. (14a), another N-soliton-like solution of Eq. (2) is

$$u=\frac{\sum_{j=1}^{N} k_j\exp[k_j x+g_i(\xi_{i1},...,\xi_{in})]}{\mu+\sum_{j=1}^{N}\exp[k_j x+g_i(\xi_{i1},...,\xi_{in})]}+u_1, \tag{36}$$

which is different from solution (25) and has the following special cases:

Case 2.1: $N = 1$
To construct a one soliton solution, $N = 1$, we take

$$\phi^{(1)} = \mu + \exp[k_1 x + g_1(\xi_{11}, \xi_{12}, ..., \xi_{1n})].$$

Under Eq. (14a), we have

(1) When $\mu = 1$, we obtain the kink-shaped soliton solution of Eq. (2)

$$u_1 = \frac{k_1}{2} \overset{\square}{} \tanh\frac{1}{2}[k_1 x + g_1(\xi_{11}, \xi_{12}, ..., \xi_{1n})]\Bigg] . \qquad (37a)$$

(2) When $\mu = -11$, we obtain the singular soliton solution of Eq. (2):

$$u_1 = \frac{k_1}{2} \overset{\square}{} \coth\frac{1}{2}[k_1 x + g_1(\xi_{11}, \xi_{12}, ..., \xi_{1n})]\Bigg] . \qquad (37b)$$

It is clearly seen that the field u_1 is not exponentially localized in all directions.
However for the two potentials $q_1 = u_x$ and $q_2 = u_y$, that is

$$q_1 = u_x = \frac{1}{4} k_1^2 sech^2 \frac{1}{2}[k_1 x + g_1(\xi_{11}, \xi_{12}, ..., \xi_{1n})], \qquad (38)$$

$$q_2 = u_y = \frac{1}{4} k_1 sech^2 \frac{1}{2}[k_1 x + f_1(\xi_{11}, \xi_{12}, ..., \xi_{1n})] \sum_{j=1}^{n} \lambda_{1j} f_{1\xi_{1j}} . \qquad (39)$$

We may find their some significant properties. In what follows we discuss the physical meanings of some special cases of the solution (38) and (39).

Case 2.1a: *Camber kink-shaped solutions*
It is easy to see from the solution (37a) that there exists a center curved surface

$$P = k_1 x + g_1(\xi_{11}, \xi_{12}, ..., \xi_{1n}) = 0. \qquad (40)$$

It is easy to know that apart from this center camber, the field u_1 exponentially tends to two different values, k_1 for $P \to +\infty$ and 0 for $P \to -\infty$. Therefore we call the solution the camber kink-shaped solution.

Case 2.1b: *Camber bell-shaped solutions*
For the solution q_1, it is easy to know that q_1 is finite on the camber (40) and decays exponentially away from the camber. Therefore we call the solution the camber bell-shaped solution.

Because $\sum_{j=1}^{n} \lambda_{1j} f_{1\xi_{1j}}$ is an arbitrary function in the solution (39), in what follows we would like to consider the abundant structures of q_2:

Case 2.1c: *Point dromion solutions*

If we set

$$F_1 = \sum_{j=1}^{n} \lambda_{1j} f_{1\xi_{1j}} = C_1 \Pi_{i=1}^{m} sech^{n_i} \xi_{1i}, \tag{41}$$

where $n_i > 0 (i = 1, ..., m)$ and C_1 is constant, then we obtain a dromion-like solution

$$q_2 = \frac{1}{4} k_1 C_1 \Pi_{i=1}^{m} sech^{n_i} \xi_{1i} sech^2 \frac{1}{2}[k_1 x + f_1(\xi_{11}, \xi_{12}, ..., \xi_{1n})], \tag{42}$$

which is obtained from an m plane soliton $sech^{n_i} \xi_{1i} (i = 1, ..., m)$ and one camber soliton.

Case 2.1d: *Ring dromion solutions*
If we set

$$F_2 = \sum_{j=1}^{n} \lambda_{1j} f_{1\xi_{1j}} = C_2 sech^n g(\xi_{11}, ..., \xi_{1m}), \tag{43}$$

where C_2 is constant and the camber $g = g(\xi_{11}, ..., \xi_{1m}) = 0$ is the surface of a cylinder, then we obtain a ring dromion solution (which means that a solution is finite on a closed curve and decays away from the curve)

$$q_2 = \frac{1}{4} k_1 C_2 sech^n g(\xi_{11}, ..., \xi_{1m}) sech^2 \frac{1}{2}[k_1 x + f_1(\xi_{11}, \xi_{12}, ..., \xi_{1n})], \tag{44}$$

which is obtained by a cylinder ($g = 0$) soliton, which is parallel to the x-axis, and a camber ($P = 0$) soliton.

Case 2.1e: *Sharp and extend dromion solutions*
Because the function f in the solution q_2 is arbitrary, the dromion-like solutions may decay much slower then an exponential or much faster than an exponential in the y–direction when the function f is replaced by different functions.

Example 1: If we take

$$F_3 = \sum_{j=1}^{n} \lambda_{1j} f_{1\xi_{1j}} = \Pi_{i=1}^{m} \left(\sum_{j=1}^{n} \sum_{s=1}^{2M} c_{ijs} \xi_{1j}^{s} \right)^{-1}, \tag{45}$$

where c_{ijs} are constants, then we obtain an extend point dromion solution (or ring dromion solution with $m = 1$)

$$q_2 = \frac{1}{4} k_1 \Pi_{i=1}^{m} \left(\sum_{j=1}^{n} \sum_{s=1}^{2M} c_{ijs} \xi_{1j}^{s} \right)^{-1} sech^2 \frac{1}{2}[k_1 x + f_1(\xi_{11}, \xi_{12}, ..., \xi_{1n})], \tag{46}$$

which decays exponentially in the x-direction and decays rationally (slower than an exponential) in the y-direction.

Example 2: If we take

$$F_4=\sum_{j=1}^{n}\lambda_{1j}f_{1\xi_{1j}}=\Pi_{i=1}^{m}sech^{n_i}\left(\sinh\sum_{j=1}^{n}\sum_{s=1}^{M}c_{ijs}\xi_{1j}^{s}\right). \quad (47)$$

where c_{ijs} are constants, then we obtain a sharp point dromion solution (or ring dromion solution with $m=1$)

$$q_2=\frac{1}{4}k_1\Pi_{i=1}^{m}sech^{n_i}\left(\sinh\sum_{j=1}^{n}\sum_{s=1}^{M}c_{ijs}\xi_{1j}^{s}\right)sech^2\frac{1}{2}[k_1x+f_1(\xi_{11},\xi_{12},...,\xi_{1n})], \quad (48)$$

which decays exponentially in the x-direction and decays much faster than an exponential in the y-direction.

Case 2.1f: *Oscillatory dromion solutions*
If we take

$$F_5=\sum_{j=1}^{n}\lambda_{1j}f_{1\xi_{1j}}=\Pi_{i=1}^{4}F_i\cos[g(\xi_{11},...,\xi_{1m})], \quad (49)$$

where $F_i(i=1,2,3,4)$ are defined by Eqs. (41), (43), (45) and (47), then we obtain oscillatory dromion solutions

$$q_2=\frac{1}{4}k_1\Pi_{i=1}^{4}F_i\cos[g(\xi_{11},...,\xi_{1m})]sech^2\frac{1}{2}[k_1x+f_1(\xi_{11},\xi_{12},...,\xi_{1n})], \quad (50)$$

which oscillates both in amplitude and in the plane.

Case 2.1g: *Singularity oscillatory dromion solutions*
If we take

$$F_6=\sum_{j=1}^{n}\lambda_{1j}f_{1\xi_{1j}}=\Pi_{i=1}^{4}F_i\tan[g(\xi_{11},...,\xi_{1m})],$$

where $F_i(i=1,2,3,4)$ are defined by Eqs. (41), (43), (45) and (47), then we obtain a singularity oscillatory dromion solutions

$$q_2=\frac{1}{4}k_1\Pi_{i=1}^{4}F_i\tan[g(\xi_{11},...,\xi_{1m})]sech^2\frac{1}{2}[k_1x+f_1(\xi_{11},\xi_{12},\cdots,\xi_{1n})],$$

which oscillates both in amplitude and in the plane.

Case 2.2: $N=2$
When $N=2$, we have a two soliton-like solution of Eq. (2)

$$u=\frac{k_1\exp[k_1x+g_1(y-(k_1+u_1)t)+k_2\exp[k_2x+g_2(y-(k_2+u_1)t)]}{1+\exp[k_1x+g_1(y-(k_1+u_1)t)]+k_2\exp[k_2x+g_2(y-(k_2+u_1)t)]}+u_1.$$

V. Conclusions

In summary, we have proved that the (2+1)-dimensional Burger equation possesses the Painlevé property by singularity analysis. Based upon the P-analysis, we have deduced a Bäcklund transformation and a Cole-Hopf transformation and they are used to find N-soliton-like solutions, infinitely many rational solutions and other new solutions. In addition, we analyze the properties of some obtained solutions. It is shown that the variable u_y admits exponentially localized solutions rather than its physical field $u(x, y, t)$. Some open questions which need further investigation are listed as follows: (1) Does Eq. (2) have other types of exact solutions? (2) Does Eq. (2) has other symmetries? (3) What is its Hamiltonian structure?

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