New Lax Integrable Hierarchies and Liouville Integrable Bi-Hamiltonian Structures Associated with an Isospectral Problem

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A new Lax integrable hierarchy associated with a properly isospectral problem with an arbitrary function, which contains the Dirac isospectral problem, is presented in this paper. As a reduction, a representative system of the generalized nonlinear Schrödinger equations in the hierarchy is explicitly given. It is shown that the Lax integrable hierarchy possesses a bi-Hamiltonian structure by using the trace identity method. In addition, it is proved that the Lax integrable hierarchy is also Liouville integrable.

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I. Introduction

As is well known, two central and important but difficult subjects in soliton theory and nonlinear integrable dynamical systems are to search for new Lax integrable hierarchies and their Hamiltonian structures [1-9]. The bi-Hamiltonian structure is of particular importance, because the demonstration of a bi-Hamiltonian structure for a system of partial differential equations is a direct and elegant method of proving its complete integrability [1-4]. Given a properly isospectral problem and its auxiliary problem

$$\phi_x = U(u;\lambda)\phi; \quad \phi_{t_n} = V^{(n)}(u;\lambda)\phi; \tag{1}$$

with λ being the spectral parameter for which $\lambda_t = 0$, the compatible condition of system (1) is evidently $\phi_{xt} = \phi_{tx}$, i.e.,

$$U_{t_n} - V^{(n)}_x + [U, V^{(n)}] = 0. \tag{2}$$

Generally speaking, system (1) is over-determined. It is interesting and important, but difficult, to search for the proper U and $V^{(n)}$ such that (2) represents a certain nonlinear Lax integrable hierarchy and to construct its bi-Hamiltonian structure

$$u_{t_n} = J\frac{\delta H_n}{\delta u} = K\frac{\delta H_{n-1}}{\delta u}; \tag{3}$$

where $u = (u_1, u_2, \ldots, u_s)$ is the potential contained in the matrix $U(u,\lambda)$ and J and K are symplectic operators, i.e., J and K are linear and skew-symmetric operators and the Jacobi equality of J and K holds as well. Also, $\delta/\delta u = (\delta/\delta u_1, \ldots, \delta/\delta u_s)$ stands for the variational derivatives

$$\frac{\delta}{\delta u_j} = \sum_{n\geq 0}(-\partial)^n\frac{\partial}{\partial u_j^{(n)}}; \quad u_j^{(n)} = \frac{\partial^n u_j}{\partial x^n}. \tag{4}$$

It is well known that the isospectral problem corresponding to the Dirac hierarchy reads [5, 6]

$$\phi_x = U\phi = U(\lambda, u)\phi; \quad U = \begin{pmatrix} q & \lambda + r \\ -\lambda + r & -q \end{pmatrix}; \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{5}$$

In this paper, we would like to discuss the isospectral problem with an arbitrary function

$$\begin{aligned} &\phi_x = U\phi = U(\lambda, u)\phi; \\ &U = \begin{pmatrix} q & \lambda + r + f(q^2 + r^2) \\ -\lambda + r - f(q^2 + r^2) & -q \end{pmatrix}; \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}; \quad u = \begin{pmatrix} q \\ r \end{pmatrix}; \end{aligned} \tag{6}$$

where q and r are two scalar potentials, λ a constant spectral parameter and $f(q^2 + r^2)$ is an arbitrary smooth function of $q^2 + r^2$.

The rest of the paper is arranged as follows: In Sec. II, a new Lax integrable hierarchy associated with (6) is obtained. In Sec. III, using the trace identity method [7-9], it is shown that the Lax integrable hierarchy is also Liouville integrable and possesses a bi-Hamiltonian structure. In Sec. IV, the first two representative systems of equations in the hierarchy are given, which contains the generalized nonlinear Schrödinger equations, and their bi-Hamiltonian structures are also given. In addition, some remarks are presented.

II. The lax integrable hierarchy associated with (6)

In order to derive a Lax integrable hierarchy associated with (6) by using the zero curvature equation, we firstly need to introduce the adjoint representation of (6)

$$V_x = [U, V] \equiv UV - VU; \quad V = V(\lambda, u) = \begin{pmatrix} a & b + c \\ b - c & -a \end{pmatrix}; \tag{7}$$

from which we easily have

$$\begin{cases} a_x = 2\lambda b - 2rc + 2bf(q^2 + r^2); \\ b_x = -2\lambda a + 2qc - 2af(q^2 + r^2); \\ c_x = 2qb - 2ra. \end{cases} \tag{8}$$

Substituting

$$a = \sum_{j \geq 0} a_j \lambda^{-j}; \quad b = \sum_{j \geq 0} b_j \lambda^{-j}; \quad c = \sum_{j \geq 0} c_j \lambda^{-j}; \tag{9}$$

(where a_j, b_j, c_j $(j = 0, 1, 2, \ldots)$ are all functions of q and r to be determined later.) into (8) leads to

$$\begin{cases} a_{n+1} = -\frac{1}{2} b_{n,x} + q c_n - f(q^2 + r^2) a_n; \\ b_{n+1} = \frac{1}{2} a_{n,x} + r c_n - f(q^2 + r^2) b_n; \\ c_{n+1} = 2\partial^{-1}(q b_{n+1} - r a_{n+1}); \end{cases} \tag{10}$$

which give rise to the following recursive formulae

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = L \begin{pmatrix} a_n \\ b_n \end{pmatrix}; \quad L = \begin{pmatrix} -2q\partial^{-1}r - f(q^2+r^2) & -\frac{1}{2}\partial + 2q\partial^{-1}q \\ \frac{1}{2}\partial - 2r\partial^{-1}r & 2r\partial^{-1}q - f(q^2+r^2) \end{pmatrix}; \tag{11}$$

$$c_{n+1} = 2\partial^{-1}(qb_{n+1} - ra_{n+1}); \quad n = 1, 2, \ldots \tag{12}$$

where $\partial = \partial/\partial x$, $\partial\partial^{-1} = \partial^{-1}\partial = 1$. If we take these initial values $a_0 = b_0 = 0$; $c_0 = \alpha = \mathrm{const.}$, then (11) and (12) imply the following results:

$$\begin{aligned} a_1 &= \alpha q; \quad b_1 = \alpha r; \quad c_1 = 0; \\ a_2 &= -\frac{1}{2}\alpha r_x - \alpha q f(q^2+r^2); \quad b_2 = \frac{1}{2}\alpha q_x - \alpha r f(q^2+r^2); \quad c_2 = \frac{1}{2}\alpha(q^2+r^2); \\ a_3 &= -\frac{1}{4}\alpha q_{xx} + \frac{1}{2}\alpha[r f(q^2+r^2)]_x + \frac{1}{2}\alpha r_x f(q^2+r^2) + \frac{1}{2}\alpha q(q^2+r^2) + \alpha q f^2(q^2+r^2); \\ b_3 &= -\frac{1}{4}\alpha r_{xx} - \frac{1}{2}\alpha[q f(q^2+r^2)]_x - \frac{1}{2}\alpha q_x f(q^2+r^2) + \frac{1}{2}\alpha r(q^2+r^2) + \alpha r f^2(q^2+r^2); \\ c_3 &= \frac{1}{2}\alpha(rq_x - qr_x) - \alpha(q^2+r^2) f(q^2+r^2); \cdots. \end{aligned} \tag{13}$$

In order to deduce a Lax hierarchy of nonlinear evolution equations from (6), we introduce the auxiliary problem of (6), namely,

$$\phi_{t_n} = V^{(n)}\phi = V^{(n)}(\lambda; u)\phi; \tag{14}$$

$$V^{(n)} = (\lambda^n V)_+ + \Delta_n = \sum_{j=0} \begin{pmatrix} a_j & b_j + c_j \\ b_j - c_j & -a_j \end{pmatrix} \lambda^{n-j} + \begin{pmatrix} 0 & \delta_{1n} \\ \delta_{2n} & 0 \end{pmatrix}; \tag{15}$$

where the symbol "+" denotes the selection of the polynomial part of λ and δ_{1n}, δ_{2n} are functions to be determined later. Therefore it is easy to prove that the compatibility condition, $\phi_{xt} = \phi_{tx}$, of the Lax pair (6) and (14) generates the zero curvature equation, $U_{t_n} - V^{(n)}_x + [U, V^{(n)}] = 0$, that is to say,

$$\delta_{1n,x} = -\delta_{2n,x} = (f(q^2+r^2))_t = 2(qq_t + rr_t) f'(q^2+r^2); \tag{16}$$

$$q_t = 2b_{n+1} + 2r\delta_{1n}; \tag{17}$$

$$r_t = -2a_{n+1} + 2q\delta_{2n} = -2a_{n+1} - 2q\delta_{1n}. \tag{18}$$

By using (16)-(18), we have

$$\begin{aligned} \delta_{1n} = -\delta_{2n} &= 2\partial^{-1}(qq_t + rr_t) f'(q^2+r^2) \\ &= 4\mu\partial^{-1} f'(q^2+r^2)(qb_{n+1} - ra_{n+1}) = 2\partial^{-1} f'(q^2+r^2) c_{n+1,x}; \end{aligned} \tag{19}$$

and the following Lax hierarchy of nonlinear evolution equations:

$$u_{t_n} = \begin{pmatrix} q_{t_n} \\ r_{t_n} \end{pmatrix} = X_n = M \begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix}; \tag{20}$$

where

$$M = \begin{pmatrix} -8r\partial^{-1}f'(q^2+r^2)r & 2+8r\partial^{-1}f'(q^2+r^2)q \\ -2+8q\partial^{-1}f'(q^2+r^2)r & -8q\partial^{-1}f'(q^2+r^2)q \end{pmatrix}.$$

By using (11), we know that (20) is equivalent to the hierarchy of nonlinear evolution equations:

$$u_{t_n} = \begin{pmatrix} q_{t_n} \\ r_{t_n} \end{pmatrix} = X_n = ML^n \begin{pmatrix} \alpha q \\ \alpha r \end{pmatrix}; \quad n = 1, 2, \cdots. \tag{21}$$

III. Bi-hamiltonian structure and liouville integrability

In this section, we would like to construct the bi-Hamiltonian structure for the hierarchy (21) and to prove that it is integrable in Liouville's sense. We first need to introduce the following new variables

$$G_{n+1} = \begin{pmatrix} G^{(1)}_{n+1} \\ G^{(2)}_{n+1} \end{pmatrix} = \begin{pmatrix} 2a_{n+1} - 4qf'(q^2+r^2)c_{n+1} \\ 2b_{n+1} - 4rf'(q^2+r^2)c_{n+1} \end{pmatrix}. \tag{22}$$

Noticing that $c_{n+1} = 2\partial^{-1}(qb_{n+1} - ra_{n+1})$, we get

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = N \begin{pmatrix} 2a_{n+1} - 4qf'(q^2+r^2)c_{n+1} \\ 2b_{n+1} - 4rf'(q^2+r^2)c_{n+1} \end{pmatrix}. \tag{23}$$

with

$$N = \begin{pmatrix} \frac{1}{2} - 2qf'(q^2+r^2)\partial^{-1}r & 2qf'(q^2+r^2)\partial^{-1}q \\ -2rf'(q^2+r^2)\partial^{-1}r & \frac{1}{2}+2rf'(q^2+r^2)\partial^{-1}q \end{pmatrix}.$$

Therefore we have the following recursive relation

$$G_{n+1} = \begin{pmatrix} 2a_{n+1} - 4qf'(q^2+r^2)c_{n+1} \\ 2b_{n+1} - 4rf'(q^2+r^2)c_{n+1} \end{pmatrix} = N^{-1}LN \begin{pmatrix} 2a_{n+1} - 4qf'(q^2+r^2)c_{n+1} \\ 2b_{n+1} - 4rf'(q^2+r^2)c_{n+1} \end{pmatrix}. \tag{24}$$

where $N^{-1} = \begin{pmatrix} \frac{1}{2}+2qf'(q^2+r^2)\partial^{-1}r & -2qf'(q^2+r^2)\partial^{-1}q \\ 2rf'(q^2+r^2)\partial^{-1}r & \frac{1}{2} - 2rf'(q^2+r^2)\partial^{-1}q \end{pmatrix}$ and $\Psi = N^{-1}LN$ is a recursive operator.

Therefore the hierarchy (21) can be rewritten as

$$u_{t_n} = \begin{pmatrix} q_{t_n} \\ r_{t_n} \end{pmatrix} = X_n = JG_{n+1} = KG_n = \cdots = \alpha J\Psi^n \begin{pmatrix} q \\ r \end{pmatrix}; \tag{25}$$

where

$$J = MN = \begin{pmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{pmatrix}; \tag{26}$$

and

$$K = MLN = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}; \tag{27}$$

with

$$\begin{aligned}
j_{11} &= -4r\partial^{-1}f'(q^2+r^2)r - 4f'(q^2+r^2)r\partial^{-1}r;\\
j_{12} &= 1 + 4f'(q^2+r^2)r\partial^{-1}q + 4r\partial^{-1}f'(q^2+r^2)q;\\
j_{21} &= -1 + 4f'(q^2+r^2)q\partial^{-1}r + 4q\partial^{-1}f'(q^2+r^2)r;\\
j_{22} &= -4f'(q^2+r^2)q\partial^{-1}q - 4q\partial^{-1}f'(q^2+r^2)q;
\end{aligned}$$

and

$$\begin{aligned}
k_{11} = \frac{1}{2}\partial &- 2r\partial^{-1}r + 2r\partial^{-1}f'(q^2+r^2)q\partial - 2\partial f'(q^2+r^2)q\partial^{-1}r\\
&+4r\partial^{-1}rf'(q^2+r^2)f(q^2+r^2) + 4rf(q^2+r^2)f'(q^2+r^2)\partial^{-1}r\\
&-8r\partial^{-1}qf'(q^2+r^2)\partial qf'(q^2+r^2)\partial^{-1}r\\
&-8r\partial^{-1}rf'(q^2+r^2)\partial rf'(q^2+r^2)\partial^{-1}r;\\
k_{12} = -&f(q^2+r^2) + 2r\partial^{-1}q + 2\partial qf'(q^2+r^2)\partial^{-1}q + 2r\partial^{-1}rf'(q^2+r^2)\partial\\
&-4r\partial^{-1}qf'(q^2+r^2)f(q^2+r^2) - 4rf'(q^2+r^2)f(q^2+r^2)\partial^{-1}q\\
&+8r\partial^{-1}qf'(q^2+r^2)\partial qf'(q^2+r^2)\partial^{-1}q\\
&+8r\partial^{-1}rf'(q^2+r^2)\partial rf'(q^2+r^2)\partial^{-1}q\\
k_{21} = f&(q^2+r^2) + 2q\partial^{-1}r - 2q\partial^{-1}qf'(q^2+r^2)\partial - 2\partial rf'(q^2+r^2)\partial^{-1}r\\
&-4qf'(q^2+r^2)f(q^2+r^2)\partial^{-1}r - 4q\partial^{-1}rf'(q^2+r^2)f(q^2+r^2)\\
&+8q\partial^{-1}qf'(q^2+r^2)\partial qf'(q^2+r^2)\partial^{-1}r + 8q\partial^{-1}rf'(q^2+r^2)\partial rf'(q^2+r^2)\partial^{-1}r;\\
k_{22} = \frac{1}{2}\partial &- 2q\partial^{-1}q - 2q\partial^{-1}f'(q^2+r^2)r\partial + 2\partial rf'(q^2+r^2)\partial^{-1}q\\
&+4q\partial^{-1}qf'(q^2+r^2)f(q^2+r^2) + 4qf(q^2+r^2)f'(q^2+r^2)\partial^{-1}q\\
&-8q\partial^{-1}qf'(q^2+r^2)\partial qf'(q^2+r^2)\partial^{-1}q\\
&-8q\partial^{-1}rf'(q^2+r^2)\partial rf'(q^2+r^2)\partial^{-1}q.
\end{aligned}$$

Proposition 1: The 2×2 matrix integro-differential operators J and K defined by (26) and (27) are Hamiltonian operators for the arbitrary function $f(q^2+r^2)$.

Proof: It is easy to see that J is a skew symmetry operator, i.e., $J^* = -J$ and J satisfies the Jacobi identity, that is to say,

$$< Z, J'(u)[JX]Y > +\text{cyclic}(X, Y, Z) \equiv 0 (\text{mod}\,\partial).$$

where $X = (X_1, X_2)^T$, $Y = (Y_1, Y_2)^T$, $Z = (Z_1, Z_2)^T$ and $J'(u)[g]$ denotes the Frechet derivative of J, i.e., $J'(u)[g] = \frac{d}{d\epsilon}J(u + \epsilon g)|_{\epsilon=0}$. Hence J is a Hamiltonian (symplectic) operator [1-3]. According to the same argument as in the above theory, we can also prove that the operator K is also a Hamiltonian(symplectic) operator.

Following the same notion applied in Refs. [8], the Killing-Cartan standard form is defined by $< X, Y >= \text{const.} \cdot \text{tr}(XY)$. For convenient calculation, we take $\text{const.} = 1$. Hence direct calculation gives

$$\begin{aligned} &< V, \frac{\partial U}{\partial q} >= 2a - 4qf'(q^2 + r^2)c, \quad < V, \frac{\partial U}{\partial r} >= 2b - 4rf'(q^2 + r^2)c, \\ &< V, \frac{\partial U}{\partial \lambda} >= -2c. \end{aligned} \tag{28}$$

According to the definition of the trace identity [7, 8], we can give the concrete form of the trace identity associated with (6)

$$\frac{\delta}{\delta u}(< V, \frac{\partial U}{\partial \lambda} >) = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}(< V, \frac{\partial U}{\partial q} >, < V, \frac{\partial U}{\partial r} >)^T, \tag{29}$$

where γ is a constant to be determined later. Substituting (9) and (28) into (29), and comparing the coefficients of λ^{-n-2} an both sides of (29) yields

$$\begin{aligned} \frac{\delta}{\delta u}(-2c_{n+2}) = (\gamma - n - 1)(&2a_{n+1} - 4qf'(q^2 + r^2)c_{n+1}, \\ &2b_{n+1} - 4rf'(q^2 + r^2)c_{n+1})^T. \end{aligned} \tag{30}$$

To fix the value of γ, we take $n = 0$ in (30) and find $\gamma(2\alpha r, 2\alpha q) = 0$, from which we get $\gamma = 0$. Therefore we have

$$G_{n+1} = \begin{pmatrix} 2a_{n+1} - 4qf'(q^2 + r^2)c_{n+1} \\ 2b_{n+1} - 4rf'(q^2 + r^2)c_{n+1} \end{pmatrix} = \frac{\delta H_n}{\delta u} = \Psi\frac{\delta H_{n-1}}{\delta u} = \Psi G_n, \tag{31}$$

with the Hamiltonian functions satisfying

$$H_{n+1} = \frac{2c_{n+2}}{n+1}, \quad n = 1, 2, \ldots . \tag{32}$$

Therefore the hierarchy of nonlinear evolution equations (21) possesses the following bi-Hamiltonian structures from (22) and (31):

$$u_{t_n} = \begin{pmatrix} q_{t_n} \\ r_{t_n} \end{pmatrix} = X_n = J\frac{\delta H_{n+1}}{\delta u} = K\frac{\delta H_n}{\delta u}. \tag{33}$$

In fact, we can easily show that the Hamiltonian functions $\{H_n\}_{n}^{\infty} = 0$ satisfying $\frac{\delta H_{n+1}}{\delta u} = \Psi\frac{\delta H_n}{\delta u} (n \geq 0)$ are all common conserved densities for the whole Lax hierarchy (21) and they commute with each other under the Poisson bracket associated with the Hamiltonian operator J. A direct calculation shows that

$$\begin{aligned} \{H_n, H_m\}_J &= \int \left\langle \frac{\delta H_n}{\delta u}, J\frac{\delta H_m}{\delta u} \right\rangle dx = \int \left\langle \frac{\delta H_n}{\delta u}, J\Psi \frac{\delta H_{m-1}}{\delta u} \right\rangle dx \\ &= \int \left\langle \frac{\delta H_n}{\delta u}, \Psi^{*} J\frac{\delta H_{m-1}}{\delta u} \right\rangle dx = \int \left\langle \Psi \frac{\delta H_n}{\delta u}, J\frac{\delta H_{m-1}}{\delta u} \right\rangle dx \\ &= \{H_{n+1}, H_{m-1}\}_J = \cdots = \{H_m, H_n\}_J, \quad n, m \geq 0. \end{aligned} \tag{34}$$

Therefore we get $\{H_n, H_m\}_J = 0$. During the course of the proof of (34), we used the following formula

$$J\Psi = MLN = K = -K^{*} = -(J\Psi)^{*} = -\Psi^{*}J^{*} = \Psi^{*}J.$$

which is obtained from proposition 1, i.e., $J^{*} = -J$ and $K^{*} = -K$. In addition we can also derive

$$[X_n, X_m] = \left[J\frac{\delta H_n}{\delta u}, J\frac{\delta H_m}{\delta u} \right] = J\frac{\delta}{\delta u}\{H_n, H_m\} = 0, \quad n, m \geq 0, \tag{35}$$

which shows that every system of the generalized Dirac equations possesses infinitely many commuting symmetries $\{X_n\}_{n=0}^{\infty}$. Thus according to the theorem in Ref. [8], we get the following proposition:

Proposition 2: (a) The Lax integrable hierarchy (21) is an integrable Hamiltonian system in Liouville's sense. (b) The Hamiltonian functions $\{H_n\}$ are conserved densities of the whole Lax integrable hierarchy (21) and they are in pairwise involution under the Poisson bracket associated with the Hamiltonian operator J.

In addition, we can directly verify

$$V_{t_n} = [V^{(n)}, V], \quad n \geq 0, \tag{36}$$

when $u_{t_n} = X_n$, i.e., $U_{t_n} - V^{(n)} + [U, V^{(n)}] = 0, n \geq 0$. In fact, we easily find that $V_{t_n} - [V^{(n)}, V]$ satisfies the adjoint representation of $\phi_x = U\phi$ and that $V_{t_n} - [V^{(n)}, V]$ vanishes at $u = 0$. Therefore (36) holds for $n \geq 0$ because the adjoint representation $V_x = [U, V]$ has uniqueness, namely, if $V_x = [U, V]$ and V vanishes at $u = 0$ vanishes, then V itself vanishes.

IV. Examples and Remarks

Example 1: Let $n = 1$, (21) reduces to a system of nonlinear evolution equations

$$\left\{ \begin{aligned} q_{t_1} &= \alpha q_x - 2\alpha r f(q^2 + r^2) + 4\alpha r \partial^{-1}(qq_x + rr_x) f'(q^2 + r^2), \\ r_{t_1} &= \alpha r_x + 2\alpha q f(q^2 + r^2) - 4\alpha q \partial^{-1}(qq_x + rr_x) f'(q^2 + r^2). \end{aligned} \right. \tag{37}$$

Example 2: Let $n = 2$, (21) becomes a new system of nonlinear evolution equations

$$\begin{aligned} q_{t_3} = & -\frac{1}{2}\alpha r_{xx} - \alpha[qf(q^2+r^2)]_x - \alpha q_x f(q^2+r^2) + \alpha r(q^2+r^2) \\ & + 2\alpha r f^2(q^2+r^2) + 2r\partial^{-1} f'(q^2+r^2)[\alpha(rq_x - qr_x) \\ & - 2\alpha(q^2+r^2)f(q^2+r^2)]_x; \end{aligned} \tag{38a}$$

$$\begin{aligned} r_{t_3} = & \frac{1}{2}\alpha q_{xx} - \alpha[rf(q^2+r^2)]_x - \alpha r_x f(q^2+r^2) - \alpha q(q^2+r^2) \\ & - 2\alpha q f^2(q^2+r^2) - 2q\partial^{-1} f'(q^2+r^2)[\alpha(rq_x - qr_x) \\ & - 2\alpha(q^2+r^2)f(q^2+r^2)]_x. \end{aligned} \tag{38b}$$

which have the bi-Hamiltonian structure

$$u_{t_3} = \begin{pmatrix} q_{t_3} \\ r_{t_3} \end{pmatrix} = X_3 = J\frac{\delta H_3}{\delta u} = K\frac{\delta H_2}{\delta u};$$

with the Hamiltonian functions

$$H_3 = \frac{1}{3}c_4; \quad H_2 = \frac{1}{2}\alpha(rq_x - qr_x) - \alpha(q^2+r^2)f(q^2+r^2).$$

Remark 1: If we take $f(q^2+r^2) = q^2+r^2$ in (38a) and (38b), (38a) and (38b) reduce to a system of generalized nonlinear Schrödinger equations

$$\left\{ \begin{aligned} q_{t_3} &= -\tfrac{1}{2}\alpha r_{xx} - 4\alpha q(qq_x + rr_x) + \alpha r(q^2+r^2) - 2\alpha r(q^2+r^2)^2; \\ r_{t_3} &= \tfrac{1}{2}\alpha q_{xx} - 4\alpha r(qq_x + rr_x) - \alpha q(q^2+r^2) + 2\alpha q(q^2+r^2)^2. \end{aligned} \right. \tag{39}$$

If we take $q = -\mathrm{Im}\,w$, $r = \mathrm{Rm}\,w$, then Eq. (39) reduces to a generalized higher-order nonlinear Schrödinger equation

$$iw_{t_3} = \frac{1}{2}\alpha w_{xx} - 2i\alpha w(|w|^2) - \alpha w|w|^2 + 2\alpha w|w|^4.$$

Remark 2: When $f(q^2+r^2) = 0$, (21) reduces to the Dirac hierarchy. Therefore the Dirac hierarchy is a special case of (21). According to the above-mentioned results, we can draw some conclusions about the Hierarchy of Dirac equations [5, 6]

$$u_{t_n} = \begin{pmatrix} q_{t_n} \\ r_{t_n} \end{pmatrix} = X_n = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = M\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = ML^nN\begin{pmatrix} \alpha q \\ \alpha r \end{pmatrix}. \tag{40}$$

where

$$L = \begin{pmatrix} -2q\partial^{-1}r & -\frac{1}{2}\partial + 2q\partial^{-1}q \\ \frac{1}{2}\partial - 2r\partial^{-1}r & 2r\partial^{-1}q \end{pmatrix}; \quad M = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}; \quad N = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

Therefore we have

$$J = MN = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \quad K = MLN = \begin{pmatrix} \frac{1}{2}\partial - 2r\partial^{-1}r & 2r\partial^{-1}q \\ 2q\partial^{-1}r & \frac{1}{2}\partial - 2q\partial^{-1}q \end{pmatrix}.$$

It is clear that J and K are both Hamiltonian operators. The hierarchy of Dirac equations (40) also possesses the following bi-Hamiltonian structures

$$u_t = \begin{pmatrix} q_t \\ r_t \end{pmatrix} = X_n = J\frac{\delta H_{n+1}}{\delta u} = K\frac{\delta H_n}{\delta u};$$

with the Hamiltonian functions satisfying $H_n = \frac{2c_{n+2}}{n+1}$, $(n = 1, 2, \ldots)$. These results are the same as the ones in Ref. [6]. They are all common conserved densities for the hierarchy of Dirac equations and commute with each other under the Poisson bracket associated with the Hamiltonian operator J. In addition, there exist many commuting symmetries $\{X_n = J\frac{\delta H_{n+1}}{\delta u}\}_{n=0}^{\infty}$ for every system of Dirac equations.

Remark 3: Many new hierarchies of nonlinear evolution equations and their Hamiltonian structures can be obtained when we choose $f(\theta) = \sin\theta, \cos\theta, \sinh\theta, \tanh\theta, e^{\pm\theta}, \ln\theta$; etc. The resulting hierarchies must have a non-polynomial form in the potentials $q; r$ and their derivatives with respect to x.

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