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A New Auto-Bäcklund Transformation and its Applications in Finding Explicit Exact Solutions for the General KdV Equation

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In this paper, based on a Lax pair of the Riccati form for the general KdV equation, a new auto-Bäcklund transformation (ABT) is presented. As an application of this ABT, since only integration is needed, a series of explicit and exact solutions can be obtained which contain soliton-like solutions. This approach is important for finding more new and physically significant exact solutions of nonlinear evolution equations.

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I. Introduction

It is well known that Lax pairs of nonlinear evolution equations play an important role in soliton theory [1, 2]. For instance, Lax pairs could be used to study symmetries of nonlinear evolution equations. In order to solve nonlinear evolution equations by use of the inverse scattering method, finding Lax pairs of nonlinear differential equations is a key step[1-4]. However, for some equations, this old method is too complex to work.

Recently, Tian *et al*. [5] presented an approach, which was obtained from systems of Lax equations, that can be used to search for exact solutions of nonlinear evolution equations. However they only considered some simple and constant coefficient nonlinear equations. They did not study variable coefficient inhomogeneous nonlinear differential equations. Recently, we considered a Bäcklund transformation and exact solutions of the variable coefficient KdV equation [6].

For the (1+1)-dimensional variable coefficient inhomogeneous KdV equation, that is, the general KdV equation [7, 8]

$$u_t - u_{xxx} - 6uu_x - 6fu + x(12f^2 - f_t) = 0, \qquad (1)$$

where $f = f(t)$ is an arbitrary function of t. Many well-known nonlinear wave equations, such as the KdV equation ($f = 0$) and the cylindrical KdV equation ($f = -\frac{1}{12t}$) [9]

$$u_t - u_{xxx} - 6uu_x + \frac{1}{2t}u = 0, \qquad (2)$$

are all special cases of (1). Nonlocal Lie-Bäcklund symmetries of (1) have been discussed [7]. In this paper, we would like to further consider the new auto-Bäcklund transformation and some exact solutions of (1).

The rest of this paper is organized as follows: In section II, we find a new Auto-Bäcklund transformation(ABT) for (1) by considering Lax pairs of (1). In section III, by using the ABT, some exact solutions for (1) are obtained, which contain solitary-like wave solutions. When using the ABT, only integration is needed. Finally, we give some conclusions and open problems.

II. Leading to a new auto-bäcklund transformation

For the given equation (1), by use of the WTC method [10] we easily show that (1) passes the Painlevè test and the Lax pair for (1) can be written as

$$\psi_{xx} = (\lambda - u + fx)\psi, \tag{3a}$$

$$\psi_t = [2(u + 2\lambda) + 4fx]\psi_x + (-u_x + f)\psi, \tag{3b}$$

with

$$\lambda_t - 12f\lambda = 0, \quad i.e. \quad \lambda(t) = c\exp\left(\int 12f\,dt\right), \tag{3c}$$

where $\lambda = \lambda(t)$ is a function of only t and c is an arbitrary constant.

In order to consider the Lax pair (3a)-(3c) further, we rewrite the above form. Introducing the following Cole-Hopf transformation,

$$\omega = \frac{\partial}{\partial x}\ln\psi = \frac{\psi_x}{\psi}, \tag{4}$$

and substituting (4) into (3a)-(3c), yields the following Lax pairs of Riccati form

$$\omega_x = \lambda + fx - u - \omega^2, \tag{5a}$$

$$\omega_t = (4fx + 2u + 4\lambda)(\lambda + fx - u) - (4fx + 2u + 4\lambda)\omega^2 + (2u_x + 4f)\omega - u_{xx}, \tag{5b}$$

or

$$\omega_x = \lambda + fx - u - \omega^2, \tag{6a}$$

$$\omega_t = (4fx + 2u + 4\lambda)\omega_x + (2u_x + 4f)\omega - u_{xx}. \tag{6b}$$

It is easily shown that the compatibility condition, $\omega_{xt} = \omega_{tx}$, of (5a) and (5b) ((6a) and (6b)) is just (1). Hence, we know that if u and ω are solutions of (5a) and (5b) ((6a) and (6b)), then u is a solution for (1). In order to seek exact solutions of (1), we only need consider the Lax pairs (5) (or (6)) of Riccati form for (1) here.

Proposition 1: Let

$$A_n(x,t) = -(4fx + 2u_n + 4\lambda)\omega_n + u_{nx} + 2f(t) + \frac{\partial}{\partial t}\left(\int \omega_n dx\right), \tag{7}$$

and

$$B_n(x,t) = -(4fx + 2u_n + 4\lambda)\exp\left(-2\int \omega_n dx\right) + 2A_n(x,t)\int \exp\left(-2\int \omega_n dx\right) dx + \frac{\partial}{\partial t}\left[\int \exp\left(-2\int \omega_n dx\right) dx\right]. \tag{8}$$

If (u_n, ω_n) satisfy (5a) and (5b), then $A_n(x,t)$ and $B_n(x,t)$ are both functions of only t, namely,

$$A_{nx}(x,t) = 0 \;\; (i.e., \;\; A_n(x,t) = A(t)); \quad B_{nx}(x,t) = 0 \;\; (i.e., \;\; B_n(x,t) = B(t)). \tag{9}$$

Proof: Using (5a) and (5b) and differentiating $A_n(x,t)$ and $B_n(x,t)$ with respect to x, respectively, it is easily shown that (9) is true.

Proposition 2: Set the following relations

$$u_{n+1} = 2\frac{\partial^2}{\partial x^2}[\ln M_n(x,t)] + u_n + 2\omega_{nx}, \tag{10a}$$

$$\omega_{n+1} = -\frac{\partial}{\partial x}[\ln M_n(x,t)] - \omega_n, \quad (n = 1,2,3,...), \tag{10b}$$

with

$$M_n(x,t) = \int \exp\left(-2\int \omega_n dx\right) dx + \exp\left[-2\int A_n(t)dt\right]\left[M_0 - \int B_n(t)\exp\left(2\int A_n(t)dt\right) dt\right]. \tag{11}$$

where M_0 is arbitrary constant. If u_n and ω_n satisfy (5a) and (5b), then u_{n+1} and ω_{n+1} also satisfy (5a) and (5b).

Proof: It is only needed to prove that u_{n+1} and ω_{n+1} satisfy (5a) and (5b) ((6a) and (6b)), namely

$$\omega_{n+1,x} = \lambda + fx - u_{n+1} - \omega_{n+1}^2, \tag{12a}$$

$$\omega_{n+1,t} = (4fx + 2u_{n+1} + 4\lambda)\omega_{n+1,x} + (2u_{n+1,x} + 4f)\omega_{n+1} - u_{n+1,xx}. \tag{12b}$$

In what follows we prove (12a) and (12b) separately.

(i) **The Proof of (12a):**
Because u_n and ω_n satisfy (6a) and (6b), namely

$$\omega_{n,x} = \lambda + fx - u_n - \omega_n^2, \tag{13a}$$

$$\omega_{n,t} = (4fx + 2u_n + 4\lambda)\omega_{n,x} + (2u_{n,x} + 4f)\omega_n - u_{n,xx}, \tag{13b}$$

we obtain

$$\omega_{n+1,x} = -\frac{\partial^2}{\partial x^2}(\ln M_n) - (\lambda + fx) + u_n + \omega_n^2; \tag{14}$$

$$\begin{aligned}\lambda + fx - u_{n+1} - \omega_{n+1}^2 &= -\frac{\partial^2}{\partial x^2}(\ln M_n) - (\lambda + fx) + u_n + \omega_n^2 \\ &\quad - \left[\frac{\partial^2}{\partial x^2}\ln M_n + 2\omega_n \frac{\partial}{\partial x}\ln M_n + \left(\frac{\partial}{\partial x}\ln M_n\right)^2\right]:\end{aligned} \tag{15}$$

From (11) we have

$$\frac{\partial}{\partial x}(\ln M_n) = \exp\left(-2\int \omega_n dx\right) M_n^{-1}; \tag{16}$$

$$\frac{\partial^2}{\partial x^2}(\ln M_n) = -2\omega_n \exp\left(-2\int \omega_n dx\right) M_n^{-1} - \exp\left(-4\int \omega_n dx\right) M_n^{-2}: \tag{17}$$

Substituting (16) and (17) into (15b) we get

$$\lambda + fx - u_{n+1} - \omega_{n+1}^2 = -\frac{\partial^2}{\partial x^2}\ln M_n - (\lambda + fx) + u_n + \omega_n^2: \tag{18}$$

Therefore, from (14) and (18), it is clearly seen that (12a) holds.

(ii) **The Proof of (12b):**
Differentiating (10b) with respect to t yields

$$\begin{aligned}\omega_{n+1,t} = &-\omega_{nt} - 2\left(\int \omega_n dx\right)_t \exp\left(-2\int \omega_n dx\right) M_n^{-1} \\ &- \exp\left(-2\int \omega_n dx\right) M_{nt} M_n^{-2}:\end{aligned} \tag{19}$$

with

$$\begin{aligned}M_{nt} = &\frac{\partial}{\partial t}\left[\int \exp\left(-2\int \omega_n dx\right) dx\right] - A_n(t)\exp\left(-\int A_n(t)dt\right) \\ &\times \left[M_0 - \int B_n(t)\exp\left(\int A_n(t)dt\right)dt\right] - B_n(t) \\ = &-A_n(t)M_n(x;t) + (4fx + 2u_n + 4\lambda)\exp\left(-2\int \omega_n dx\right):\end{aligned} \tag{20}$$

Substituting (20) into (19) and combining the condition that u_n and ω_n satisfy (5b), i.e., (13a) and (13b) yields

$$\begin{aligned}\omega_{n+1,t} = &u_{nxx} - (4f + 2u_{nx})\left[\omega_n + \exp\left(-2\int \omega_n dt\right) M_n^{-1}\right] - (4fx + 2u_n + 4\lambda) \\ &\left[\omega_{nx} - 2\omega_n \exp\left(-2\int \omega_n dx\right) M_n^{-1} - \exp\left(-4\int \omega_n dx\right) M_n^{-2}\right]:\end{aligned} \tag{21}$$

According to (10b) and (16), (21) can be rewritten as follows

$$\omega_{n+1;t} = u_{n;xx} + (4f + 2u_{nx})\omega_{n+1} + (4fx + 2u_n + 4\lambda)\omega_{n+1;x}. \tag{22}$$

From (10a) and (12a) which has been proved, we easily get the following equations

$$\begin{aligned} u_n &= u_{n+1} + 2\omega_{n+1;x} = u_{n+1} + 2(\lambda + fx - u_{i+1} - \omega_{i+1}^2) \\ &= 2(\lambda + fx - \omega_{i+1}^2) - u_{n+1}; \end{aligned} \tag{23}$$

$$\begin{aligned} u_{nx} &= -u_{n+1;x} - 4\omega_{n+1}\omega_{n+1;x} + 2f \\ &= -u_{n+1;x} - 4\omega_{n+1}(\lambda + fx - u_{n+1} - \omega_{n+1}^2) + 2f; \end{aligned} \tag{24}$$

$$\begin{aligned} u_{nxx} &= -u_{n+1;xx} + 4\omega_{n+1}u_{n+1;x} - 4f\omega_{n+1} \\ &\quad - 4(\lambda + fx - u_{n+1} - 3\omega_{n+1}^2)\omega_{n+1;x}. \end{aligned} \tag{25}$$

Substituting (23)-(25) into (22), we can obtain (12b), that is to say, (12b) is also true. Hence we have completed the proof of this proposition.

According to proposition 2, we have

Proposition 3: (10a) and (10b) form a new Bäcklund transformation for (1). Where $u_n; \omega_n$ satisfy (5a) and (5b) ((6a)and (6b)) and M_n satisfies (11) as well as $\lambda_t - 12f\lambda = 0$, i.e., $\lambda = c\exp(\int 12f dt)$.

III. Applications of the ABT and explicit exact solutions

For a solution $(u_n; \omega_n)$ of (5) or (6), only integration is needed, then we can get another explicit exact solution $(u_{n+1}; \omega_{n+1})$ of (5) or (6) by using the ABT (10a) and (10b). According to the same procedure, we can also obtain the third solution $(u_{n+2}; \omega_{n+2})$ of (5) or (6), and so on. Thus it is easy to see that $u_n; u_{n+1}; u_{n+2}; \ldots$ in these conclusions are just exact solutions of (1). In what follows we choose the following cases to illustrate the ABT.

Case 1 Take

$$u_1 = \lambda + xf = c\exp\left(\int 12f dt\right) + xf; \quad \omega_1 = 0. \tag{26}$$

It is clear that $(u_1; \omega_1)$ is a solution of (6), thus we get

$$\int \omega_1 dx = g(t); \qquad \int \exp\left(-2\int \omega_1 dx\right) dx = \exp[-2g(t)]x + h(t). \tag{27}$$

Here $g(t)$ and $h(t)$ are both integral functions of t. Thus according to the definitions of $A_1(t); B_1(t)$ and $M_1(x;t)$, we have

$$\begin{aligned} A_1(t) &= g_t + 3f; \quad B_1(t) = h_t + 2hg_t + 3hf - 6c\exp\left(-2g + \int 12f dt\right); \\ M_1(x;t) &= \exp(-2g)x + h + \exp\left(-2g - 6\int f dt\right) \\ &\quad \left\{M_0 - \int\left[h_t + 2hg_t + 3hf - 6c\exp\left(-2g + \int 12f dt\right)\right]\exp\left(2g + 6\int f dt\right)dt\right\}. \end{aligned} \tag{28}$$

Finally, according to the ABT (10a) and (10b), we obtain

$$
\begin{aligned}
u_2 = {} & xf + c\exp\left(\int 12 f\,dt\right) - 2\exp(-4g)\Big\{\exp(-2g)x + h \\
& + \exp\left(-2g - 6\int f\,dt\right) \times \Big[M_0 - \int \Big[h_t + 2hg_t + 3hf - 6c\exp\Big(-2g \\
& + \int 12 f\,dt\Big)\Big] \exp\left(2g + 6\int f\,dt\right) dt\Big]\Big\}^{-2}. \\
\omega_2 = {} & -\exp(-2g)\Big\{\exp(-2g)x + h + \exp\left(-2g - 6\int f\,dt\right) \\
& \Big[M_0 - \int [h_t + 2hg_t + 3hf - 6c\exp\Big(-2g + \int 12 f\,dt\Big)\Big] \\
& \exp\left(2g + 6\int f\,dt\right) dt\Big]\Big\}^{-1}.
\end{aligned} \tag{29}
$$

where $f = f(t)$, $g = g(t)$ and $h = h(t)$. It is clear that u_2 is a rational fractional solution for (1). From the solution (u_2, ω_2) of (6) again, by virtue of the auto-Bäcklund transformation (10), we can also derive another exact solution of (1). But the formal solution is rather complicated, we omit it here.

Case 2 Take another solution of (6), namely,

$$
u_1(x,t) = xf + (c - \lambda^2)\exp\left(\int 12 f\,dt\right), \quad \omega_1 = \lambda\exp\left(\int 6 f\,dt\right). \tag{30}
$$

where $\lambda \neq 0$ is an arbitrary constant. Then by use of (30), we have

$$
\begin{aligned}
& \int \omega_1 dx = \lambda\exp\left(\int 6ft\right) x + k_1(t), \\
& \int \exp\left(-2\int \omega_1 dx\right) dx = -\frac{1}{2\lambda}\exp\left[-2\lambda x e^{\int 6f dt} - \int 6f\,dt - 2k_1(t)\right] + k_2(t),
\end{aligned} \tag{31}
$$

where $k_1(t)$, $k_2(t)$ are integral functions of t. Thus we obtain

$$
A_1(t) = (6c - \lambda^2)e^{\int 18 f dt} + 3f + k_{1t}(t), \quad B_1(t) = 0. \tag{32}
$$

Substituting $A_1(t), B_1(t)$ into $M_1(x,t)$ yields

$$
\begin{aligned}
M_1(x,t) = {} & -\frac{1}{2\lambda}\exp\left[-2\lambda x e^{\int 6f dt} - \int 6f\,dt - 2k_1(t)\right] + k_2(t) \\
& + M_0\exp\left[-2(6c - \lambda^2)\int e^{\int 18 f dt} dt + \int 3f\,dt + k_1(t)\right].
\end{aligned} \tag{33}
$$

Finally, via use of the Bäcklund transformation (10), we can obtain

$$
\begin{aligned}
& u_2(x,t) = \frac{2(M_{1xx}M_1 - M_{1x}^2)}{M_1^2} + xf + (c - \lambda^2)\exp\left(\int 12 f\,dt\right), \\
& \omega_2(x,t) = -\frac{M_{1x}}{M_1} - \lambda\exp\left(\int 6 f\,dt\right).
\end{aligned} \tag{34}
$$

In fact, $u_2(x,t)$ is a soliton-like solution for (1), which can be rewritten as follows.

(a) When $\mathrm{sgn}(\mu) = -\mathrm{sgn}(k_2(t) + M_0 \exp[-2(6c-\mu^2)\int e^{\int 18 f\,dt}dt + \int 3f\,dt + k_1(t)])$, we get a bell-type soliton-like solution for (1),

$$\begin{aligned} u(x,t) = {} & xf + (c-\mu^2)\exp\left(-\int 12 f\,dt\right) + 2\mu^2 \exp\left(-12\int f\,dt\right)\mathrm{sech}^2\Big\{-\mu x e^{\int 6f\,dt} \\ & - \int 3f\,dt - k_1(t) - \frac{1}{2}\ln\Big[-2\mu k_2(t) - 2\mu M_0 \exp\Big(-2(6c-\mu^2)\int e^{\int 18 f\,dt}dt \\ & + \int 3f\,dt + k_1(t)\Big)\Big]\Big\}. \end{aligned} \tag{35}$$

(b) When $\mathrm{sgn}(\mu) = \mathrm{sgn}(k_2(t) + M_0 \exp[-2(6c-\mu^2)\int e^{\int 18 f\,dt}dt + \int 3f\,dt + k_1(t)])$, we get a singular soliton-like solution for (1),

$$\begin{aligned} u(x,t) = {} & xf + (c-\mu^2)\exp\left(-\int 12 f\,dt\right) + 2\mu^2 \exp\left(-12\int f\,dt\right)\mathrm{csch}^2\Big\{-\mu x e^{\int 6f\,dt} \\ & - \int 3f\,dt - k_1(t) - \frac{1}{2}\ln\Big[2\mu k_2(t) + 2\mu M_0 \exp\Big(-2(6c-\mu^2)\int e^{\int 18 f\,dt}dt \\ & + \int 3f\,dt + k_1(t)\Big)\Big]\Big\}. \end{aligned} \tag{36}$$

IV. Conclusions

In summary, we have derived a new auto-Bäcklund transformation (ABT) for the variable coefficient inhomogeneous KdV equation (VCIKdVE) by virtue of a Lax pair of the VCIKdVE. Based on the ABT, three types of exact solutions are obtained, which may be important for explaining some physical phenomena. This approach may be extended to other systems of nonlinear differential equations and higher dimensional nonlinear wave equations, Lax pairs may be applied to find other properties, such as symmetries, conservation laws, and so on. These problems need to be studied further.

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