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The Hofstadter Problem

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We give in this article a new discussion of the Hofstadter problem. We use the algebraic methods of conventional quantum mechanics to get the Bethe-Ansatz equation, which was previously obtained via quantum group methods. The algebraic calculations give a new derivation of the Chambers formula.

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I. Introduction

The problem of an electron in a constant magnetic field is an interesting academic subject. Classically the electron moves in a circle in a plane perpendicular to the magnetic field. The quantum mechanical problem was solved by Landau [1] who obtained the wavefunction of the problem in a certain gauge known as the Landau gauge. The energy levels came to be called the Landau levels. The wavefunction depends on the gauge as well as the boundary conditions. The wavefunction for the symmetric gauge was obtained by Laughlin [2]. The discretised version of the Landau problem was investigated by Hofstadter [3]. This became known as the Hofstadter problem. The problem was studied for the case of a rational flux. Periodicity reduces the Schröedinger difference equation to a one-dimensional problem. The energy eigenvalues can be obtained numerically showing a butterfly pattern. It was shown by Wiegmann and Zabrodin [4] that the Hofstadter problem for rational flux possesses quantum group $U_q(sl_2)$ symmetries and the energy eigenvalues at mid-band can be determined by a Bethe-Ansatz equation. Hence the Hofstadter problem is an integrable system.

The appearance of quantum group symmetries in the Hofstadter problem is quite subtle. It would be nice pedagogically to re-examine the problem by the conventional algebraic methods of quantum mechanics. The present paper is an attempt in such a direction. This paper is organised as follows. Section 2 is a recapitulation of the Hofstadter problem in its original form. In the Landau gauge we get an eigenvalue difference equation, the Harper equation. We consider the case of rational flux denoted by the quantity Q. Pedagogically we solve the eigenvalue equation for low Q. The eigenvalues exhibit a cyclic symmetry but this behaviour is not generic. Section 3 is a new derivation of the Chambers formula [5] by algebraic means. The Chambers formula is the key formula determining the spectrum of the Hofstadter problem. In Section 4 we give a discussion of the quantum group structure of the problem as in [4]. In section 5 we use the conventional algebraic methods of quantum mechanics to study the Hofstadter problem. It is shown that the same Bethe-Ansatz equation is obtained. In the last section we give our discussion and conclusion.

II. The Hofstadter problem

We consider an electron hopping on a two-dimensional square lattice in a uniform magnetic field. The hamiltonian is given by

$$H = \sum_{\langle n,m\rangle} e^{iA_{\vec{n},\vec{m}}} c^{\dagger}_{\vec{m}} c_{\vec{n}}, \tag{1}$$

where $\langle n,m\rangle$ denote nearest neighbours, $c^{\dagger}_{\vec{n}}$ and $c_{\vec{n}}$ are creation and annihilation operators, respectively for the electron Wannier wavefunction. The flux per plaquette is denoted by Φ such that

$$\prod_{\text{plaquette}} e^{iA_{\vec{n},\vec{m}}} = e^{i\Phi}. \tag{2}$$

We are going to study the case when $\Phi = 2\pi\frac{P}{Q}$ where P and Q are relatively prime integers.

The hamiltonian depends on the choice of a gauge. The most popular Landau gauge is to adopt

$$A_{\vec{n},\vec{n}+\hat{x}} = 0, \qquad A_{\vec{n},\vec{n}+\hat{y}} = n_x\Phi. \tag{3}$$

In this gauge we have a plane wave along the y-axis and along the x-axis we have a Bloch wavefunction with periodicity Q.

$$\psi(\vec{n}) = e^{ik_y n_y} \times e^{ik_x n_x} u_{n_x}(\vec{k}), \tag{4}$$

where

$$u_{n_x+Q} = u_{n_x}. \tag{5}$$

Thus, in this gauge the hamiltonian is separable and we obtain a one-dimensional difference equation

$$e^{ik_x} u_{n+1} + e^{-ik_x} u_{n-1} + \left(e^{i(k_y+n\Phi)} + e^{-i(k_y+n\Phi)} \right) u_n = E u_n, \tag{6}$$

where E is the energy eigenvalue. This is just a $Q \times Q$ matrix eigenvalue problem for the hamiltonian

$$\begin{pmatrix} \omega e^{ik_y} + \omega^{-1} e^{-ik_y} & e^{ik_x} & \cdots & e^{-ik_x} \\ e^{-ik_x} & \omega^2 e^{ik_y} + \omega^{-2} e^{-ik_y} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ e^{ik_x} & 0 & \cdots & e^{ik_y} + e^{-ik_y} \end{pmatrix}, \tag{7}$$

where $\omega = e^{i\Phi}$. To solve for E we have to solve a polynomial of degree Q.

It is not hard to show that the polynomial of degree Q determining E is of the form [5]

$$p(E) = 2(\cos Qk_x + \cos Qk_y). \tag{8}$$

That is, E depends only on $\Lambda = 2(\cos Qk_x + \cos Qk_y)$ and for each value of Λ we get Q E's. As Λ varies we get Q energy bands.

It is instructive to list here some of the polynomials for low Q,

$$Q = 2; \qquad E^2 - 4 = \Lambda; \tag{9}$$

$$Q = 3; \qquad E^3 - 6E = \Lambda; \tag{10}$$

$$Q = 4; \qquad E^4 - 8E^2 + 4 = \Lambda. \tag{11}$$

Hofstadter [3] found the range of E for all Λ and different Q by numerical calculations. He found the famous butterfly pattern for the spectra. Indeed, it is easy to write down the solutions of E for low Q.

For $Q = 2$, we have the solutions

$$E_0 = 2\rho\cos\theta; \quad E_1 = 2\rho\cos(\theta + \pi); \tag{12}$$

where

$$\rho = \sqrt{2}; \text{ and } e^{i2\theta} + e^{-i2\theta} = \frac{\Lambda}{2}. \tag{13}$$

For $Q = 3$, we have the solutions

$$E_0 = 2\rho\cos\theta; \quad E_1 = 2\rho\cos\left(\theta + \frac{2\pi}{3}\right); \quad E_2 = 2\rho\cos\left(\theta + \frac{4\pi}{3}\right); \tag{14}$$

where

$$\rho = \sqrt{2}; \text{ and } e^{i3\theta} + e^{-i3\theta} = \frac{\Lambda}{\sqrt{2^3}}. \tag{15}$$

For $Q = 4$, we have the solutions

$$E_n = 2\rho\cos\left(\theta + \frac{2n\pi}{4}\right); \qquad n = 0, 1, 2, 3; \tag{16}$$

where

$$\rho = \sqrt{2}; \text{ and } e^{i4\theta} + e^{-i4\theta} = \frac{4 + \Lambda}{\sqrt{2^4}}. \tag{17}$$

These patterns are not generic since the spectra for higher Q do not have such symmetries.

We have another useful picture called the Floquet picture where we denote

$$\phi_n = e^{ink_x} u_n; \tag{18}$$

and we can write the difference equation as

$$\psi_{n+1} + \psi_{n-1} + \left(\omega^n e^{ik_y} + \omega^{-n} e^{-ik_y}\right)\psi_n = E\psi_n; \tag{19}$$

with the boundary conditions

$$\psi_{n+Q} = e^{iQk_x}\psi_n. \tag{20}$$

The difference equation can be written as a transfer matrix equation

$$\begin{pmatrix} \psi_{n+1} \\ \psi_n \end{pmatrix} = \begin{pmatrix} E - (\omega^n e^{ik_y} + \omega^{-n} e^{-ik_y}) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_n \\ \psi_{n-1} \end{pmatrix}; \tag{21}$$

and the polynomial for the eigenvalue problem can be written in an equivalent form

$$\mathrm{Tr}\left(\prod_{n=1,\ldots,Q} \begin{pmatrix} E - (\omega^n e^{ik_y} + \omega^{-n} e^{-ik_y}) & -1 \\ 1 & 0 \end{pmatrix} \right) = e^{iQk_x} + e^{-iQk_x}. \tag{22}$$

III. The Chambers formula

The Chambers formula is the key equation for generating the spectrum of the Hofstadter problem. So it is profitable to examine this equation in more detail. Indeed, originally Chambers [5] discussed this formula in the context of a network model, that is, he employed the Floquet picture to interpret this problem. Another proof along similar lines was given by Wilkinson [6]. We are going to provide a proof of this formula in a more algebraic way. We think that this proof is more transparent and reflects the noncommutative nature of the magnetic translation operators [7]. The hamiltonian can be written in an operator form

$$H = T_x + T_y + T_{-x} + T_{-y}; \tag{23}$$

where the operators T_x, T_y satisfy the commutation relation

$$T_x T_y = \omega T_y T_x. \tag{24}$$

The flux is rational so that ω^Q is unity. In this section we do not work with any concrete representation.

The key ingredient of our proof is to observe that the hamiltonian to the Q-th power is expressed as

$$(T_x + T_y + T_{-x} + T_{-y})^Q = T_x^Q + T_y^Q + T_{-x}^Q + T_{-y}^Q + \cdots. \tag{25}$$

Now it is easy to verify that T_x^Q, T_y^Q and their conjugates commute with each other and with the hamiltonian. Indeed, the operators T_x^Q, T_y^Q and their conjugates are proportional to the identity operator so that we can write

$$T_x^Q = e^{iQk_x}; \quad \text{and} \quad T_y^Q = e^{iQk_y}. \tag{26}$$

So H^Q and $T_x^Q + T_y^Q + T_{-x}^Q + T_{-y}^Q$ have simultaneous eigenvalues E^Q and ϵ, respectively. The remaining eigenvalues can only be eigenvalues of H only and the possible combination is

$$\epsilon + a_{Q-2}E^{Q-2} + \cdots = E^Q; \tag{27}$$

which is just the Chambers formula.

Indeed we can do better by deriving the possible form of the polynomials. We are going to manipulate products of the sum $T_x + T_y$. The first useful little formula is

$$(T_x + T_y)^n = \sum_{l=0} \begin{bmatrix} n \\ l \end{bmatrix} T_y^{n-l} T_x^l; \tag{28}$$

where

$$\begin{bmatrix} n \\ l \end{bmatrix} = \frac{(1-\omega^n)(1-\omega^{n-1})\cdots(1-\omega^{n-l+1})}{(1-\omega)(1-\omega^2)\cdots(1-\omega^l)}. \tag{29}$$

The expression $\begin{bmatrix} n \\ l \end{bmatrix}$ is called the gaussian polynomial and was known to Gauss long ago in the form of the expansion

$$\prod_{i=0}^{n-1}(1 + \omega^i x) = \sum_{l} \begin{bmatrix} n \\ l \end{bmatrix} \omega^{l(l+1)/2} x^l. \tag{30}$$

It is remarkable that for $n = Q$ the formula simplifies to

$$(T_x + T_y)^Q = T_x^Q + T_y^Q. \tag{31}$$

The procedure becomes more complicated if we want to find the hamiltonian to the Q-th power. Recursively we can find

$$\begin{aligned} (T_x + T_y + T_{-x} + T_{-y})^n &= (T_x + T_y)^n + (T_{-x} + T_{-y})^n + (T_x + T_y)^{n-2} \\ &\quad \times \sum_{i=0}^{n-1} \left[2 + \omega^{-i} T_y T_{-x} + \omega^{i} T_x T_{-y} \right] + \cdots. \end{aligned} \tag{32}$$

The other expressions are quite complicated. In the case $n = Q$ we have the simpler expression which yields the Chambers formula as

$$E^Q = \epsilon + 2QE^{Q-2} + a_{Q-4}E^{Q-4} + \cdots. \tag{33}$$

It is a spectacular feat of Wiegmann and Zabrodin [4] to demonstrate that the exact spectra can be obtained by the Bethe Ansatz. They found that there is a quantum group symmetry in this problem. We shall exploit these ideas in the next section.

IV. Magnetic translations and the quantum group

It is well-known that an electron in a constant magnetic field exhibits magnetic translational invariance [7]. In our problem, in the Landau gauge and the Bloch picture, we have the following two operators T_x and T_y explicitly as

$$T_x = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix} e^{ik_x}; \tag{34}$$

and

$$T_y = \begin{pmatrix} \omega & 0 & \cdots & 0 & 0 \\ 0 & \omega^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \omega^{Q-1} & 0 \\ 0 & 0 & \cdots & 0 & \omega^Q \end{pmatrix} e^{ik_y}. \tag{35}$$

It is interesting to note that such a symmetry first appeared in connection with integrable models in [8]. The two operators have the commutation relation

$$T_xT_y = \omega T_yT_x. \tag{36}$$

The phase factor just reflects the gauge invariance of the problem. The hamiltonian for the Hofstadter problem can be decomposed as

$$H = T_x + T_{-x} + T_y + T_{-y}. \tag{37}$$

It should be noted that T_x^Q and T_y^Q commute with T_x and T_y simultaneously because of the periodic boundary conditions and hence with the hamiltonian H.

On the other hand there is the quantum group theory studied extensively in the eighties [9, 10]. Relevant to our discussion is the algebra $U_q(sl_2)$ (a q-deformation of the universal enveloping algebra of sl_2) generated by the elements A, B, C, D with the commutation relations

$$\begin{aligned} AB &= qBA; & BD &= qDB; \\ DC &= qCD; & CA &= qAC; \\ AD &= 1; & [B, C] &= \frac{A^2 - D^2}{q - q^{-1}}. \end{aligned} \tag{38}$$

The deformation parameter is $q = e^h$. As $q \to 1$ it gives the classical sl_2 algebra : $(A^2 - D^2)/(q - q^{-1}) \to S_3$, $B \to S_+$, $C \to S_-$. The centre of this algebra is the Casimir operator

$$c = \left(\frac{q^{-\frac{1}{2}}A - q^{\frac{1}{2}}D}{q - q^{-1}} \right)^2 + BC. \tag{39}$$

There is a cyclic $(2j+1)$-dimensional finite representation of the algebra

$$A = \begin{pmatrix} q^{j} & 0 & \cdots & 0 & 0 \\ 0 & q^{j-1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & q^{-j+1} & 0 \\ 0 & 0 & \cdots & 0 & q^{-j} \end{pmatrix}; \tag{40}$$

$$B = \begin{pmatrix} 0 & [1]_q & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & [2j]_q \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}; \tag{41}$$

and

$$C = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ [2j]_q & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & [1]_q & 0 \end{pmatrix}; \tag{42}$$

where $[n]_q = (q^n - q^{-n})/(q - q^{-1})$ and $q^{2j+1} = \pm 1$. In this representation the value of the central element c is

$$c = \left(\frac{q^{j+\frac{1}{2}} - q^{-j-\frac{1}{2}}}{q - q^{-1}} \right)^2 = [j+\tfrac{1}{2}]_q^2 . \tag{43}$$

Now we can observe that the operators T_x, T_y and the generators of the quantum group $U_q(sl_2)$ are somewhat related. First of all we must take $(2j+1) = Q$. The following correspondence can be checked to be true.

$$\begin{aligned} &T_{-y}T_x = \pm qA^2; \quad T_{-x}T_y = \pm q^{-1}D^2; \\ &T_x + T_y = i(q - q^{-1})B; \\ &T_{-x} + T_{-y} = \pm i(q - q^{-1})C; \end{aligned} \tag{44}$$

with

$$T_x T_y = q^2 T_y T_x ; \tag{45}$$

where we have $q = e^{i\frac{\Phi}{2}}$, whence $q^Q = -1$ for P odd and $q^Q = 1$ for P even. The $\pm$ signs correspond to P being odd and even, respectively, since then the central element will have the correct value of $-4(q - q^{-1})^2$ for P odd and 0 for P even. In the following discussion we shall adopt P to be odd. Explicitly, with the present representation for A, B, C and D, we can solve for T_x and T_y

$$T_x = -i \begin{pmatrix} 0 & q^{-1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & q^{-2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & q^{-(Q-1)} \\ q^{-Q} & 0 & 0 & \ddots & 0 & 0 \end{pmatrix}; \tag{46}$$

$$T_y = i \begin{pmatrix} 0 & q & 0 & \cdots & 0 & 0 \\ 0 & 0 & q^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & q^{(Q-1)} \\ q^{Q} & 0 & 0 & \ddots & 0 & 0 \end{pmatrix}. \tag{47}$$

There is some freedom in choosing the correspondence of the operators T_x, T_y and the elements of the quantum group. Wiegmann and Zabrodin [4] discovered that the above identification corresponds to $\Lambda = 0$ in the chiral gauge. Indeed, if we observe that $B^Q = 0$, implying $e^{iQk_x} + e^{iQk_y} = 0$. Thus the Hofstadter hamiltonian for the mid-band point can be identified as

$$H = i(q - q^{-1})(B + C); \tag{48}$$

and the difference equation can be written as a functional equation

$$i(z^{-1} + qz)\Psi(qz) - i(z^{-1} + q^{-1}z)\Psi(q^{-1}z) = E\Psi(z). \tag{49}$$

Now we know that $\Psi(z)$ is a polynomial of degree $Q - 1$,

$$\Psi(z) = \prod_{m=1}^{Q-1}(z - z_m). \tag{50}$$

With this ansatz, the functional equation can be written as

$$i(z^{-1} + qz)\prod_{m=1}^{Q-1}\frac{qz - z_m}{z - z_m} - i(z^{-1} + q^{-1}z)\prod_{m=1}^{Q-1}\frac{q^{-1}z - z_m}{z - z_m} = E. \tag{51}$$

The left-hand side of this equation is a meromorphic function, whereas the right-hand side is a constant. The residues must all vanish. The residues at $z = 0$ and $z = \infty$ are zeroes without further restriction. The zeroes of the residues at $z = z_m$ give the Bethe Ansatz equation,

$$\frac{q + z_l^2}{1 + qz_l^2} = \prod_{m=1; m\neq l}^{Q-1}\frac{qz_l - z_m}{z_l - qz_m}; \quad l = 1, \ldots, Q - 1; \tag{52}$$

and E is given by the sum of the roots z_m

$$E = -i(q - q^{-1}) \sum_{m=1}^{Q-1} z_m . \tag{53}$$

Since E is real, thus the z_m must be real or come in complex conjugate pairs. Also it is easy to see that E and $-E$ are both solutions. This verifies that this is the spectra for the mid-band point, $\theta = 0$. It may be useful to investigate the symmetries hidden in the Bethe Ansatz equation.

V. Algebraic methods for the eigenvalue problem

We reexamine the eigenvalue problem here using purely algebraic methods. This is to demystify the Bethe-ansatz like equations proposed by Wiegmann and Zabrodin [4]. To proceed, we start with the hamiltonian

$$H = T_x + T_y + T_{-x} + T_{-y} ; \tag{54}$$

in purely algebraic form. We are going to take the basis with the operator $T_{-y}T_x$ diagonalised. Thus we take any eigenvector

$$T_{-y}T_x|u\rangle = u|u\rangle . \tag{55}$$

Since $T_{-y}T_x$ is unitary, u is a phase and part of it can be absorbed into the wavefunction. We shall specify u later. The operators $T_x + T_y$ and $T_{-x} + T_{-y}$ act like the rotation operators,

$$(T_x + T_y)|u\rangle \propto |\omega u\rangle ; \tag{56}$$

$$(T_{-x} + T_{-y})|u\rangle \propto |\omega^{-1}u\rangle . \tag{57}$$

Since we have

$$(T_{-x} + T_{-y})(T_x + T_y) = 2 + T_{-y}T_x + T_{-x}T_y . \tag{58}$$

We can determine the coefficients for the rotation up to a phase by

$$(T_x + T_y)|u\rangle = \left(u^{\frac{1}{2}} + u^{-\frac{1}{2}}\right)|\omega u\rangle ; \tag{59}$$

and

$$(T_{-x} + T_{-y})|u\rangle = \left(q^{-1}u^{\frac{1}{2}} + qu^{-\frac{1}{2}}\right)|\omega^{-1}u\rangle ; \tag{60}$$

where $q = \omega^{\frac{1}{2}}$. We can choose a value u_0 for u. Hence $|u_0\rangle$, $|\omega u_0\rangle$, $\ldots$, $|\omega^{Q-1}u_0\rangle$ form a complete orthonormal basis.

The value of u_0 is related to the parameter θ which we defined previously. Indeed, the operator $(T_x + T_y)^Q$ is a diagonal matrix proportional to the identity. The midband point problem,

$\Lambda = 0$, considered by Wiegmann and Zabrodin [4] is especially simple. This implies that $T_x + T_y$ is a nilpotent matrix and the value of u_0 is determined to be

$$u_0 = -\omega . \tag{61}$$

Hence, for the midband point we have the basis

$$|k\rangle = |-\omega^{k+1}\rangle ; \qquad k = 0, \ldots , Q-1 . \tag{62}$$

and

$$(T_x + T_y)|k\rangle = i\left(q^{(k+1)} - q^{-(k+1)}\right)|k+1\rangle ; \tag{63}$$

$$(T_{-x} + T_{-y})|k\rangle = i\left(q^{k} - q^{-k}\right)|k-1\rangle . \tag{64}$$

We now expand our energy eigenvectors in terms of our basis vectors as

$$|E\rangle = \sum_{k=0}^{Q-1} c_k|k\rangle . \tag{65}$$

The coefficients c_k determine the eigenvalues of the problem. It is useful to express these coefficients as the symmetric functions of the roots of the polynomial

$$\begin{aligned} \Psi(z) &= \prod_{m=1}^{Q-1}(z - z_m) \\ &= \sum_{k=0}^{Q-1} c_k z^k ; \end{aligned} \tag{66}$$

since the action of $(T_x + T_y)$ and $(T_{-x} + T_{-y})$ on $\Psi(z)$ is especially simple. We adopt the normalisation that c_{Q-1} is unity.

$$(T_x + T_y)\Psi(z) = iz\left[q\Psi(qz) - q^{-1}\Psi(q^{-1}z)\right] ; \tag{67}$$

$$(T_{-x} + T_{-y})\Psi(z) = iz^{-1}\left[\Psi(qz) - \Psi(q^{-1}z)\right] . \tag{68}$$

By the same manipulation we get the Bethe-Ansatz equation Eq. (52). It is easy to see that the energy eigenvalue E is given by

$$E = i\left[q - q^{-1}\right]c_{Q-2} . \tag{69}$$

For odd Q, obviously there is a zero energy solution. The zero energy eigenvector can be readily written down. The coefficients c_k's are zero for odd k and

$$c_{Q-n} = -\frac{q^{n+1} - q^{-(n+1)}}{q^{n} - q^{-n}}c_{Q-n-2} ; \quad \text{for } n \text{ odd} . \tag{70}$$

Hatsugai, Kohmoto and Wu [11] observed that in this case the roots of the Bethe-Ansatz equation are given by

$$z_m = iq^{2m-\frac{1}{2}}, iq^{-2m+\frac{1}{2}}, \quad m = 1, \ldots, (Q-1)/2. \tag{71}$$

VI. Discussion and conclusion

In this article we have given another viewpoint for studying the Hofstadter problem. We would like to emphasise that purely algebraic manipulations give more insight into this problem. It can be seen that the present derivation of the Chambers formula is more cogent than the previous arguments. In short, the algebraic manipulations here can be more accessible to the average layman. This shows that quantum group symmetries are not essential for the Bethe-Ansatz equation. Of course a deeper understanding of these quantum group symmetries is still waiting for us to explore. A few points should be noted. The cyclic permutation symmetry of the energy eigenvalues is not generic. However, for specific cases this symmetry is very elegant. It would be nice to investigate under what conditions this symmetry is manifest. Also for certain values of Q the problem of study should possess fermionic behaviour. Such fermionic representations should be an interesting subject.

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