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Chemical Equilibrium of a Degenerate Gas of Nucleons and Electrons in a Strong Magnetic Field

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The equations that define the equilibrium of a homogeneous relativistic gas of neutrons, protons and electrons in a constant magnetic field are obtained. We compute the relative densities of the particles at equilibrium as function of the density of neutrons and the magnetic field. It is found that, when only the first Landau level is being filled, the proton density is enhanced as compared to the case without the magnetic field. For an ultrastrong field there exists the possibility that the proton density is greater than the neutron density. However, when higher Landau levels are being filled, the values of the particle density at equilibrium quickly converge to those obtained without the magnetic field.

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The discovery of giant magnetic fields at the surface of neutron stars has greatly stimulated recent interest in the behavior of relativistic electrons in a constant strong magnetic field [1, 2, 3]. Observations of hard X-rays from pulsar Hercules X-1 allowed one to estimate the magnetic field at the pulsar surface to be of the order of 10^{13} G. Such a magnetic field "frozen" in a neutron star must become stronger and stronger, reaching an ultrastrong value of the order of 10^{18} G in the central part of the neutron star [4]. It is also believed that a new class of gamma-ray bursters, so-called soft gamma repeaters [5], are newly born neutron stars that have very large surface magnetic fields of the order up to 10^{15} G. These extremely magnetized neutron stars have been named the magnetars [6]. Furthermore, fields of the order of 10^{23} G [7] and even possibly 10^{33} G [8] may exist at the electroweak phase transition. It is worthwhile noting that the magnetic field in a neutron star may be considered as a macroscopically uniform field with respect to the characteristic scales, such as the Compton wavelengths, of constituent particles of the neutron star. However, the field is extremely nonuniform at the electroweak phase transition. In any case, such a strong field will affect in an essential way the behavior of charged particles in the star.

Inspired by these developments, we shall consider in this paper the effects of strong constant magnetic fields on the chemical equilibrium of a degenerate ideal gas of neutrons (n), protons (p), and electrons (e). We assume in what follows that the gas is spatially homogeneous due to the homogeneity of the magnetic field.

The first physical quantity which we must define is the chemical potential μ_i $(i = e, p, n)$, or the Fermi energy (if the gas temperature is equal to zero), of the relativistic charged and neutral particles as a function of magnetic field.

Without any magnetic field ($B = 0$) and at zero temperature ($\theta = 0$), the chemical potential μ_i of an ideal gas of particles of type i and mass m_i is related to the particle density n_i by [9]

$$(\tilde{\mu}_i^2 - 1)^{3/2} = n_i/n_{0i}\ ; \quad n_{0i} = m_i^3 c^3/3\pi^2\hbar^3 . \tag{1}$$

In the presence of a magnetic field $\mathbf{B} = (0, 0, B)$, the chemical potential and density of the neutron gas are still given by Eq. (1). But now the number density and chemical potential of the charged particles are related by

$$n_i = \frac{eB}{4\pi^2\hbar^2 c}\sum_n (2 - \delta_{0n})\int_{-\infty}^{\infty}\frac{dp}{\exp((E_n - \mu_i)/\theta) + 1}\ ; \quad i = e, p\ ; \tag{2}$$

where $e > 0$ is the magnitude of the elementary charge, $E_n = (p^2c^2 + m_i^2c^4 + 2eB\hbar cn)^{1/2}$ is the energy spectrum of a relativistic charged particle, $n = 0, 1, 2, \ldots$ enumerates the Landau levels, p is the momentum component parallel to $\mathbf{B}$, and θ is the temperature. In what follows we shall express values of μ_i in dimensionless units: $\tilde{\mu}_i = \mu_i/mc^2$. Eq. (2) may be considered as a relation between n_i, μ_i and θ at a given B. By integrating with respect to p in (2), making allowance for $E_{max} = \mu_i$ at $\theta = 0$, it is easy to obtain:

$$\frac{\sqrt{2}n_i}{3n_{0i}}\left(\frac{B_{0i}}{B}\right)^{3/2} = \left((\tilde{\mu}_i^2 - 1)\frac{B_{0i}}{2B}\right)^{1/2} + 2\sum_{n=1}^{n_{mi}}\left((\tilde{\mu}_i^2 - 1)\frac{B_{0i}}{2B} - n\right)^{1/2} . \tag{3}$$

Here the critical field $B_{0i} = m_i^2c^3/e\hbar$, and the value n_{mi} is given by $n_{mi} = [(\tilde{\mu}_i^2 - 1)B_{0i}/2B]$, where $[x]$ is the integral part of x. We note here that $B_{0e} = 4.41 \cdot 10^{13}$ G and $B_{0p} = 3.4 \cdot 10^6\ B_{0e}$.

Let us now consider the conditions for chemical equilibrium of a degenerate gas of protons, neutrons and electrons in the presence of large magnetic fields. We also suppose that the temperature θ is equal to zero, since for a typical 100-year old neutron star, its temperature is estimated to be 10^8 K (about 10 keV), which can be considered cold as compared to the Fermi energy (about 1000 MeV) of the degenerate relativistic neutrons [10].

We are interested in reactions in which the total density of baryons $n_b = n_p + n_n$ is conserved and the charge neutrality condition of a gas $n_p = n_e$ is satisfied . These processes are called the direct URCA processes [11, 9, 10]. Our purpose here is to determine the density of protons as a function of the density of neutrons from the condition of chemical equilibrium, namely, $\mu_n = \mu_p + \mu_e$, and the neutrality condition. A similar problem but in the context of the equation of state, *i.e.* pressure of the gas as a function of the particle density, was considered in [12] for magnetic fields much smaller than B_{0p}, namely, for B up to only $100B_{0e} \approx 10^{-4}B_{0p}$. For ultrastrong magnetic fields ($B > B_{0p}$), the problem was first considered in [13]. In [13], however, the problem was only partially solved by restricting to the lowest Landau level of electrons and/or protons only. Also, for $B < B_{0p}$ the chemical potential of the protons was approximated by the expression obtained in the absence of the magnetic field. Here we shall determine the density of protons numerically without making any approximation.

An essential observation which we employed in our numerical solution is that, under the charge neutrality condition $n_p = n_e$, the chemical potentials μ_p and μ_e in the presence of a magnetic field, as given by Eq. (3), are related by

$$\tilde{\mu}_p = \left[1 + \left(\tilde{\mu}_e^2 - 1\right)\cdot\left(\frac{m_e}{m_p}\right)^2\right]^{1/2} ; \tag{4}$$

which is exactly the relation satisfied by μ_p and μ_e without a magnetic field. In the absence of a magnetic field, Eq. (4) is easily proved from Eq. (1). To show that Eq. (4) is also satisfied by the chemical potentials when $B \neq 0$, we substitute $n_p = n_e$, $B_{0p}/B_{0e} = (m_p/m_e)^2$ and $n_{0p}/n_{0e} = (m_p/m_e)^3$ into Eq. (3) for the proton ($i = p$), we get

$$\begin{aligned} \frac{\sqrt{2}n_e}{3n_{0e}}\left(\frac{B_{0e}}{B}\right)^{3/2} &= \left[(\mu_p^2 - 1)(m_p/m_e)^2\frac{B_{0e}}{2B}\right]^{1/2} \\ &+ 2\sum_{n=1}^{n_{mp}}\left((\mu_p^2 - 1)(m_p/m_e)^2\frac{B_{0e}}{2B} - n\right)^{1/2} . \end{aligned} \tag{5}$$

The left hand side of Eq. (5) exactly equals the left hand side of Eq. (3) for the electron ($i = e$). Since Eq. (3) is a monotonic increasing function, we get by equating the right hand side of both Eq. (3) and (5) the equality $n_{mp} = n_{me}$ and

$$(\mu_p^2 - 1)(m_p/m_e)^2 = (\mu_e^2 - 1). \tag{6}$$

Hence Eq. (4) is proved. The equality $n_{mp} = n_{me}$ implies that the Landau levels of electrons and protons, are populated in the same manner. This is a direct consequence of the equality of the number of electrons and protons assumed here and the fact that the density of states per Landau level is independent of the mass of the charged particles.

Our numerical results are presented in Fig. 1 and 2 for strong ($B < B_{0p}$) and ultrastrong ($B > B_{0p}$) magnetic fields, respectively. These curves represent the normalized proton density number n_p/n_{0n} as a function of the normalized neutron density number n_n/n_{0n} at various values of magnetic field. The dotted curve gives the corresponding values in the absence of the magnetic field. In the presence of the magnetic field, the density of protons is a piecewise continuous and monotonic increasing function of the neutron density. The values of the proton density at which the first derivatives do not exist are the values at which a Landau level is being completely filled. It turns out that, for $B < B_{0p}$, the approximation made in [13] by treating the protons as if no magnetic field were present is a very good one.

From the figures one observes the following distinctive effects that a strong magnetic field has on the chemical equilibrium of the degenerate gas of nucleons and electrons: (a) when only the first Landau level is populated, the values of n_p/n_{0n} in the presence of finite B's are for the most part higher than the corresponding value when the field is absent; (b) in the presence of an ultrastrong magnetic field, there appear values of densities (populating only the first Landau level) for which $n_p > n_n$; and (c) when higher Landau levels are being filled, the curves corresponding to finite magnetic fields all converge to the one without magnetic field.

Features (a) and (b) had been noted previously in [13]. Such behavior may be understood as follows. When the neutron density at a fixed magnetic field is low enough (or equivalently, when the magnetic field at a fixed neutron density is high enough) that only the first Landau level is filled by the electrons and protons, the chemical potentials of the electrons and protons are given by (from Eq. (1) and (3)):

$$\begin{aligned} \mu_n &= m_n c^2\left[1 + (n_n/n_{0n})^{2/3}\right]^{1/2} ; \\ \mu_i &= m_i c^2\left[1 + (2n_i B_{0i})^2/(3n_{0i}B)^2\right]^{1/2} ; \quad i = e, p . \end{aligned} \tag{7}$$

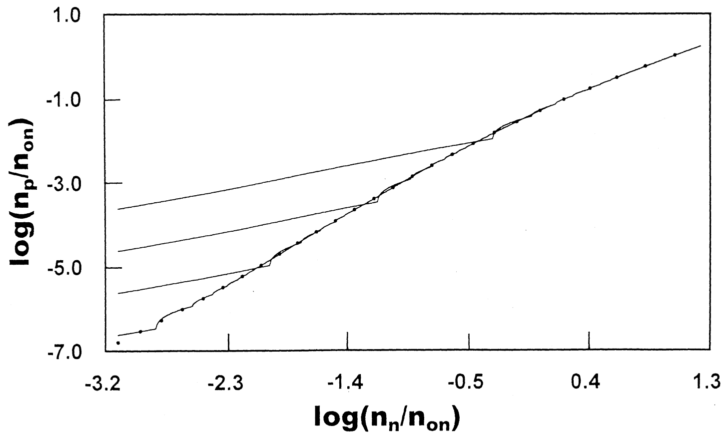

FIG. 1. Plot of $\log(n_p/n_{0n})$ verses $\log(n_n/n_{0n})$ (continuous curves) for values of B/B_{0e} at (from bottom to top): 10^2, 10^3, 10^4 and 10^5. The dotted line corresponds to the curve with $B = 0$.

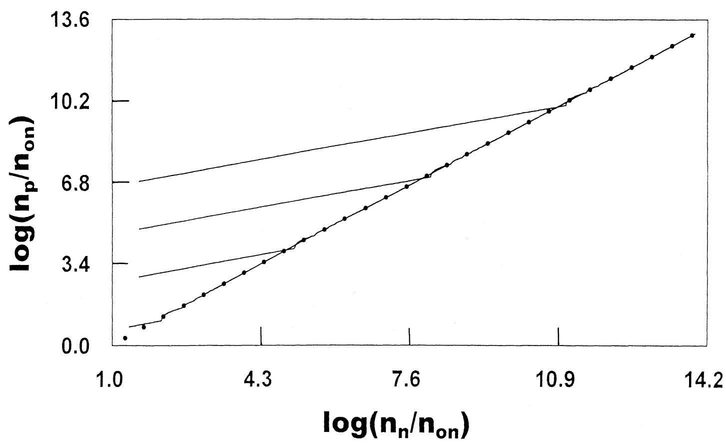

FIG. 2. Same as in Fig. 1, but for values of B/B_{0e} at (from bottom to top): 10^7, 10^9, 10^{11} and 10^{13}.

Eq. (7) shows that a higher magnetic field B tends to lower the the chemical potential of the electron when $B > B_{0e}$ (and the chemical potential of the proton as well when $B > B_{0p}$). To maintain chemical equilibrium among the particles at a fixed value of n_n/n_{0n}, the chemical potentials of the electrons and protons have to be raised so that the equilibrium condition $\mu_n = \mu_p + \mu_e$ is still satisfied. This is achieved through the increase in the density of electrons, and hence the density of protons by the neutrality condition assumed here. For a given value of n_n/n_{0n}, the value of n_p/n_{0n} in the presence of finite B's are, for the most part, higher than the corresponding value when the field is absent, until the proton density reaches a value $(n_p/n_{0n})_{cross} = (3B/2B_0)^{3/2}\,(m/m_n)^3$, which is the point of intersection between the curves with $B \neq 0$ and $B = 0$. When $B > B_{0p}$, there exist values of the densities for which $n_p > n_n$. This gives the possibility of stars with proton-rich matter. These ranges of particle densities are of great interest, since in zero magnetic field the ratio n_p/n_n is always less than unity, with a maximum equal to 1/8 [9]. The fact that the density of proton is raised under these conditions has the important implication that the direct URCA reaction is enhanced, leading to a more efficient neutron star cooling through neutrino emission.

However, as soon as the higher Laudau levels are being filled, the behaviour of the system under the magnetic field does not deviate much from that without the magnetic field. In fact, for large n they are nearly identical. This can be proved as follows. For large n we have $n_{mi} \approx (\mu_i^2 - 1)B_{0i}/2B$. Then Eq. (3) can be approximated by

$$\begin{aligned} \frac{\sqrt{2}n_i}{3n_{0i}}\left(\frac{B_{0i}}{B}\right)^{3/2} &\approx 2\int_0^{n_{mi}} (n_{mi} - n)^{1/2}\, dn \\ &= \frac{4}{3}\left[\mu_i^2 - 1\right]^{3/2}\left(\frac{B_{0i}}{2B}\right)^{3/2}. \end{aligned} \tag{8}$$

This leads to $n_i/n_{0i} = (\mu_i^2 - 1)^{3/2}$, which is exactly the result, namely Eq. (1), when $B = 0$.

To conclude, we have discussed the effects of strong constant magnetic fields on the chemical equilibrium of a degenerate gas of neutrons, protons, and electrons. The equations for chemical equilibrium under the condition of charge neutrality are numerically solved for for different values of the magnetic field and the neutron density. The effect of higher Landau levels were considered, which was left out in previous work [13]. It is found that the chemical potentials of the electrons and the protons are related in the same way, namely, through Eq. (4), regardless of whether there is a magnetic field or not. When only the first Landau level is being filled, the proton density is enhanced, as compared to the case without the magnetic field. For an ultrastrong field there exists the possibility that the proton density is greater than the neutron density, giving a proton-rich matter. However, when higher Landau levels are being filled, the values of the particle density at equilibrium quickly converge to those obtained without the magnetic field. The latter result indicates that, when the density of the particles are very high, the influence of the magnetic field is negligible.

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