Cocycles, Phases and Magnetic Translations

M. K. Fung
Department of Physics, National Taiwan Normal University,
Taipei, Taiwan 116, R.O.C.
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We demonstrate that the proper way of discussing translation invariance in the problem of a free electron in a constant magnetic field is through the magnetic translation group. Phases occurring in different energy eigenfunctions due to the magnetic translations are studied, and how the magnetic translation operators transform the energy eigenstates within the same Landau level is worked out.

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I. Introduction

The mathematical framework of quantum mechanics is based on the concept of a Hilbert space. Wavefunctions in the Hilbert space can admit a complex phase. Cocycles [1] are mathematical tools to study such phases. The occurrence of a 2-cocycle signifies the noncommutativity of the transformation operators and at the same time there is a phase in the representation space of the wavefunctions so that its coboundary is nonzero. An electron in a constant magnetic field exhibits such a phenomenon. In this problem the translational invariance should be replaced by magnetic translational invariance. The magnetic translation operators have an additional term due to a contribution from the magnetic field. We study in this paper the consequences of such a phase.

This paper is organised as follows. In section 2 we review the theory of the uses of cocycles in quantum mechanics with some classical examples. In section 3 we discuss the magnetic translational invariance of the problem of an electron in a constant magnetic field. In section 4 we study the effects of this phase for different gauges and boundary conditions of the Landau problem of a free electron in a constant magnetic field. In section 5 we study the lattice version of the Landau problem. Finally in section 6 we give our conclusion and discussion.

II. Cocycles in physics

Cocycles are mathematical tools to investigate phases in the representation theory of linear operators. Quantum mechanics is concerned with operators acting on physical states. It has been known for some time that cocycles are present in the quantum mechanics of point particles [1]. Modern gauge field theory also makes use of them [2] in the investigation of the anomaly phenomena. We give here some classical examples of 1-cocycles and 2-cocycles. We want to give a new discussion of cocycles in the Landau problem in parallel with these well-known examples.

Consider the quantum mechanics of a free particle. Translational invariance is an obvious symmetry of this problem. The translation transformation is implemented by the unitary operator

$$U(a) = e^{ia\cdot p}. \tag{1}$$

We work in units where $\hbar = c = 1$. Using the canonical commutation relations we can establish the action of $U(a)$ on operators r and $\dot{r}$ as follows

$$U(a)rU^{-1}(a) = r + a; \tag{2}$$

$$U(a)\dot{r}U^{-1}(a) = \dot{r}. \tag{3}$$

We emphasise that the present situation is simple because p is a constant of the motion.

To discuss the action of the unitary operator on the wavefunction we have to introduce a representation. Usually we work in the co-ordinate representation where the co-ordinate operator is diagonal and the momentum operator is represented as

$$p \to \frac{1}{i}\nabla. \tag{4}$$

Thus the translation transformation acting on the wavefunction $\psi(r)$ gives

$$e^{ia\cdot p}\psi(r) = \psi(r + a). \tag{5}$$

The above is the simplest realisation of the transformation group. We can add phases in the representation theory and we shall elaborate these in what follows. These phases are called cocycles. The classical example is a 1-cocycle for the Galilean boost operator. In quantum mechanics we can represent a Galilean boost operator as

$$U(v) = e^{iv\cdot(pt-mr)}. \tag{6}$$

Let us note that the bracketed term in the exponent is a constant of motion for the free particle and is proportional to the position of the particle at $t = 0$. The boost operator has the correct transformation rules for r and $\dot{r}$,

$$U(v)rU^{-1}(v) = r + vt; \tag{7}$$

$$U(v)\dot{r}U^{-1}(v) = \dot{r} + v. \tag{8}$$

However, the operator $U(v)$ acting on the wavefunction gives

$$e^{iv\cdot(pt-mr)}\psi(r) = e^{-i\omega_1(r;v)}\psi(r + vt); \tag{9}$$

where the phase ω_1 is given by

$$\omega_1(r; v) = mv\cdot r + \frac{1}{2}mv^2t. \tag{10}$$

Such a phase is called a 1-cochain. We can define the coboundary of a 1-cochain as

$$\Delta\omega_1(q,g_1,g_2) \equiv \omega_1(q^{g_1},g_2) - \omega_1(q,g_{12}) + \omega_1(q,g_1), \tag{11}$$

where q is the variable acted upon by members of the transformation group labelled by g and g_1g_2 is abbreviated as g_{12}. A 1-cochain with vanishing coboundary is called a 1-cocycle. It can easily be verified that ω_1 of Eq. (10) is indeed a 1-cocycle. This just reflects the fact that the composition law of two boosts does not require a phase,

$$U(v_1)U(v_2) = U(v_1 + v_2). \tag{12}$$

Now, the composition of a boost and a translation is not commutative on the quantum level, although they are commutative in the classical case [3],

$$U(v)U(a) = e^{-i\omega_2(r,v,a)}U(a)U(v), \tag{13}$$

where the phase ω_2 is given by

$$\omega_2(r,v,a) = -mv.a. \tag{14}$$

This implies that we have an anomalous commutator. The phase ω_2 of Eq. (14) is a 2-cochain. Indeed, it is a 2-cocycle since its coboundary vanishes. The coboundary for a 2-cocycle is defined as

$$\Delta\omega_2 \equiv \omega_2(q^{g_1},g_2,g_3) - \omega_2(q,g_{12},g_3) + \omega_2(q,g_1,g_{23}) - \omega_2(q,g_1,g_2). \tag{15}$$

The vanishing of the coboundary of a 2-cocycle ensures that the U's associate quantum mechanically.

The anomalous commutator introduces a 1-cochain whose coboundary does not vanish. In our example of Galilean invariance we can combine a boost and a translation together. There are two distinct ways of ordering. In both cases we get a phase when they operate on the wavefunctions. One way is illustrated by the formula,

$$e^{iv.(pt-mr)}e^{ia.p}\psi(r) = e^{-i\omega_1}\psi(r + a + vt). \tag{16}$$

The phase is the same as before, due to the special ordering of the two operators. It is a 1-cocycle. Or, we can change the ordering, getting

$$e^{ia.p}e^{iv.(pt-mr)}\psi(r) = e^{-i\omega_1'}\psi(r + vt + a), \tag{17}$$

where the 1-cochain ω_1' is given by

$$\omega_1'(r,(a,v)) = mv.(r + a) + \frac{1}{2}mv^2t. \tag{18}$$

This 1-cochain has nonvanishing coboundary [4] which is just the 2-cocycle of Eq. (14). Its nontriviality just reflects the fact that the commutator is anomalous.

III. Cocycles for the magnetic translation

We now switch to the problem of a free electron of charge e in a constant magnetic field B. Due to the presence of the magnetic field the discussion of translational invariance is a bit complicated. Indeed, as discussed in [5] the system possesses translational invariance, provided that we take into account of the contribution of the field momentum from the magnetic field. We can introduce the magnetic translation operators k_x and k_y [6, 7] of the form

$$k_x = m\dot{x} - eBy; \tag{19}$$

$$k_y = m\dot{y} + eBx. \tag{20}$$

Due to the presence of the magnetic field the momenta and the velocities are not proportional to each other.

$$p_x = m\dot{x} + eA_x; \tag{21}$$

$$p_y = m\dot{y} + eA_y; \tag{22}$$

where A is the vector potential for the magnetic field. The co-ordinates and conjugate momenta obey the canonical commutation relations. However, the velocities have gauge independent commutation relations,

$$[x^i, \dot{x}^j] = \frac{i}{m}\delta^{ij}; \tag{23}$$

$$[\dot{x}^i, \dot{x}^j] = \frac{ieB}{m^2}\epsilon^{ij}; \qquad i, j = 1, 2. \tag{24}$$

The magnetic translation operators do not commute,

$$[k_x, k_y] = -ieB. \tag{25}$$

It can be easily proved that k_x and k_y commute with the velocities and hence they commute with the hamiltonian, since it is a sum of squares of velocities. Thus k_x and k_y are true constants of motion. They generate translation transformations in the presence of a magnetic field. So we have two magnetic translation operators defined by

$$T_x(a) = e^{iak_x}; \tag{26}$$

$$T_y(b) = e^{ibk_y}. \tag{27}$$

Now the two operators do not commute,

$$T_y(b)T_x(a) = e^{-i\omega_2}T_x(a)T_y(b); \tag{28}$$

where the phase ω_2 is given by

$$\omega_2((x, y); b, a) = eBba. \tag{29}$$

It can be easily verified that this phase satisfies the vanishing coboundary condition. So it is a 2-cocycle. This implies that there is no breaking of associativity for this projective representation.

To discuss how these operators act on the wavefunction we must adopt a representation. Again, the co-ordinate representation is the most convenient one. Besides, we must take a gauge so that the representation takes meaning for the velocities. We first adopt the Landau gauge,

$$A_x = -By, \tag{30}$$

$$A_y = 0, \tag{31}$$

and

$$k_x = p_x, \tag{32}$$

$$k_y = p_y + eBx, \tag{33}$$

so that translation along the x-axis does not produce any phase,

$$T_x(a)\psi(x,y) = \psi(x+a,y). \tag{34}$$

The translation along the y-axis does introduce a phase

$$T_y(b)\psi(x,y) = e^{ieBbx}\psi(x,y+b). \tag{35}$$

This phase of Eq. (35) can be easily verified to be a 1-cocycle. The non-commutativity of $T_x(a)$ and $T_y(b)$ together with the phase in Eq. (35) give rise to a 1-cochain with nonvanishing coboundary,

$$T_y(b)T_x(a)\psi(x,y) = e^{-i\omega_1'}\psi(x+a,y+b), \tag{36}$$

where

$$\omega_1'((x,y),(b,a)) = -eBb(x+a). \tag{37}$$

The coboundary of this 1-cochain is nonzero,

$$\Delta\omega_1((x,y),(b_1,a_1),(b_2,a_2)) = eBb_1a_2, \tag{38}$$

and this is just the phase for the noncommutativity of the two operators in Eq. (28).

We can also work in the symmetric gauge,

$$A_x = -\frac{By}{2}, \tag{39}$$

$$A_y = \frac{Bx}{2}, \tag{40}$$

and

$$k_x = p_x - \frac{eBy}{2}, \tag{41}$$

$$k_y = p_y + \frac{eBx}{2}. \tag{42}$$

So translations along the two directions both introduce a phase,

$$T_x(a)\Psi(x,y) = e^{-\frac{ieBay}{2}}\Psi(x+a,y), \tag{43}$$

$$T_y(b)\Psi(x,y) = e^{\frac{ieBbx}{2}}\Psi(x,y+b). \tag{44}$$

Again, we have

$$T_y(b)T_x(a) = e^{-ieBba}T_x(a)T_y(b). \tag{45}$$

IV. Magnetic translations and the energy eigenfunctions

In the previous section we have discussed the action of the magnetic translation operators on general wavefunctions with appropriate phases. The Landau problem has been studied extensively and complete energy eigenfunctions have been obtained for different situations. It would be nice to see how the magnetic translation operators acts on the energy eigenfunction. Indeed, the magnetic translation operators commute with the hamiltonian. The problem is nontrivial since we have a large degeneracy of states. The magnetic translation operators will transform wavefunctions within the same Landau level. We shall study the ground state wavefunctions only, so as not to make things too complicated. The first instance is the solution first obtained by Landau for a free electron in a transverse magnetic field on a plane. In this case we adopt the Landau gauge. The energy wavefunction is a plane wavefunction in one direction and a harmonic oscillator wavefunction in the other direction. The good quantum numbers are κ and the energy E where κ is the eigenvalue of the operator k_x. The wavefunction is represented as

$$\psi_{n,\kappa}(x,y) = e^{i\kappa x}H_n(y-\kappa), \tag{46}$$

where H_n are the Hermite polynomials of the harmonic oscillator and we have set the cyclotron frequency ω as well as the mass m of the electron to unity to simplify our expression for the wavefunction. The number n denotes the n^{th} Landau level. The ground state wavefunction is

$$\psi_{0,\kappa}(x,y) = Ce^{i\kappa x}e^{-(y-\kappa)^2/2}, \tag{47}$$

where C is a normalisation constant.

Let us examine how the two magnetic translation operators act on the ground state wavefunction. Since k_x and k_y commute with the hamiltonian, these operators transform the wavefunctions within the Landau level. The k_x operator gives

$$T_x(a)\psi_{0,\kappa}(x,y) = e^{i\kappa a}\psi_{0,\kappa}(x,y). \tag{48}$$

This just introduces an unimportant phase and the energy wavefunction is unchanged by this transformation.

The operation of the k_y operator is more interesting. The k_y operator acting on the wavefunction $\psi_{0,\kappa}$ gives

$$T_y(b)\psi_{0,\kappa}(x,y) = Ce^{-ibx}e^{i\kappa x}e^{-(y-\kappa+b)^2/2}, \tag{49}$$

$$= \psi_{0,\kappa-b}(x,y). \tag{50}$$

The projective representation is just good enough to enable the k_y operator to transform the wavefunction within the Landau level to another with $\kappa - b$ as its eigenvalue for k_x.

For two successive operations we have, on one hand, the expression

$$T_x(a)T_y(b)\psi_{0,\kappa}(x,y) = e^{i(\kappa-b)a}\psi_{0,\kappa-b}(x,y), \tag{51}$$

and on the other hand the other expression

$$T_y(b)T_x(a)\psi_{0,\kappa}(x,y) = e^{i\kappa a}\psi_{0,\kappa-b}(x,y). \tag{52}$$

The wavefunctions characterised by κ are overcomplete if we have a finite plane. It is natural to propose that we confine the magnetic translations to be on a lattice with fundamental lengths a, b so that $ab = 2\pi$. Then the magnetic translation operators commute on this lattice, and the states generated will form a complete set.

Next we can consider problem of an electron moving on a 2-torus of unit dimensions with a magnetic field normal to the torus. In Manton [8] the magnetic field is chosen to be $B = 2\pi$ so that the total magnetic flux is 2π, the same flux as is emitted by an elementary Dirac monopole. This is contrived to make the multiplicity of the Landau level to be exactly 1. The cyclotron frequency in this case is 2π. The ground state wavefunction is

$$\begin{aligned}\psi(x,y) &= C\sum_{n=-\infty}^{\infty} e^{-\pi(n-y)^2+2\pi inx} \\ &= C\,\Theta(y+ix,1),\end{aligned} \tag{53}$$

where Θ is the third Jacobi theta function, and C is a normalisation constant.

Now the magnetic translation operators acting on the ground state wavefunction give

$$T_x(a)\psi(x,y) = C\sum_{n=-\infty}^{\infty} e^{-\pi(n-y)^2+2\pi in(x+a)}, \tag{54}$$

$$T_y(b)\psi(x,y) = Ce^{-i2\pi bx}\sum_{n=-\infty}^{\infty} e^{-\pi(n-y-b)^2+2\pi inx}. \tag{55}$$

Magnetic translations by unit lengths give the identity operation,

$$T_x(1)\psi(x,y) = \psi(x,y), \tag{56}$$

$$T_y(1)\psi(x,y) = \psi(x,y). \tag{57}$$

This gives a clean argument of the translation invariance of the periodic boundary conditions. The traditional way of argument is the imposition of a pure translation of the boundary conditions plus a global gauge transformation [9].

The last instance we will examine is the solution of the Landau problem in the symmetric gauge by Laughlin [10] in which explicit rotational invariance is kept. The lowest Landau level wavefunctions are

$$\psi_m(x;y) = C_m(x - iy)^m e^{-(x^2+y^2)/4}; \tag{58}$$

where m is a non-negative integer. Now the wavefunction is changed under the magnetic translations as

$$T_x(a)\psi_m(x;y) = C_m e^{-a^2/4} e^{-a(x-iy)/2}(x - iy + a)^m e^{-(x^2+y^2)/4}; \tag{59}$$

$$T_y(b)\psi_m(x;y) = C_m e^{-b^2/4} e^{-ib(x-iy)/2}(x - iy + ib)^m e^{-(x^2+y^2)/4}; \tag{60}$$

which generate a linear combination of all the states in the lowest Landau level. This complication is understandable as translation symmetry is not apparent in an explicitly rotational symmetric wavefunction. Rather we can consider the coherent state wavefunctions for the lowest Landau level [11]

$$\phi_\xi = e^{-(\alpha^2+\beta^2)} e^{(-\beta+i\alpha)(x-iy)} \left(\frac{e^{-(x^2+y^2)/4}}{\sqrt{2\pi}}\right); \tag{61}$$

where $\xi = \sqrt{2}(-\beta + i\alpha)$. The magnetic translation operators acting on this coherent wavefunction give

$$T_x(a)\phi_\xi = e^{i\alpha a}\phi_{\xi'}; \tag{62}$$

$$T_y(b)\phi_\xi = e^{i\beta b}\phi_{\xi''}; \tag{63}$$

where

$$\xi' = \sqrt{2}(-\beta - a/2 + i\alpha); \tag{64}$$

$$\xi'' = \sqrt{2}(-\beta + i\alpha - ia/2). \tag{65}$$

The phase factors in the transformation just reflect the fact that $T_x(a)$ and $T_y(b)$ do not commute.

V. Bloch electron in a magnetic field

In this section we consider putting the Landau problem on a lattice. The underlying lattice structure introduces a periodicity competing with the periodicity of the phase due to the magnetic flux. The hamiltonian of the so called Hofstadter problem [12] is given by

$$H = \hat{\tau}_x + \hat{\tau}_{-x} + \hat{\tau}_y + \hat{\tau}_{-y}; \tag{66}$$

where $\hat{T}_x$ is the covariant translation operator in the x-direction by unit lattice spacing and $\hat{T}_y$ the covariant translation operator in the y-direction by unit lattice spacing. The incommensurate case will give rise to the Harper equation [13], which is quite complicated. Here, we consider the commensurate case where the flux through a unit plaquette is given by

$$|e|Bab = 2\pi \frac{P}{Q}, \tag{67}$$

where P, Q are relatively prime integers and now a, b are the basic lattice spacing along the x and y-directions. We put

$$\omega = e^{2\pi i P/Q}, \tag{68}$$

and ω is a Q-root of unity.

The basic magnetic translation operators for unit lattice spacing are represented as

$$T_x = T_x(a), \tag{69}$$

$$T_y = T_y(b). \tag{70}$$

We shall adopt the Landau gauge. The energy wavefunction $\psi_{n,m}$ on the lattice is given by

$$\psi_{n,m} = e^{i(k_x n + k_y m)} u_m, \tag{71}$$

where k_x and k_y are the pseudo-momenta and $u_{m+Q} = u_m$. This form is in conformity with the Bloch conditions

$$T_x \psi_{n,m} = C \psi_{n,m}, \tag{72}$$

$$T_y^Q \psi_{n,m} = C' \psi_{n,m}. \tag{73}$$

It is easily seen that the energy wavefunction is also an eigenfunction of T_x and T_y^Q with eigenvalues e^{ik_x} and e^{iQk_y}, respectively. Indeed, we have the Chambers formula [14] for the energy,

$$f(E) = e^{iQk_x} + e^{-iQk_x} + e^{iQk_y} + e^{-iQk_y}, \tag{74}$$

where $f(E)$ is a polynomial of degree Q.

Recently it is found that [15] the magnetic translation operators can form a representation of the quantum group algebra $U_q(sl_2)$. It is tempting to examine how the phases are associated with these quantum group operators. We did look into the problem. But the energy wavefunctions do not transform nicely under these quantum group operators. Maybe quantum group symmetries will be manifest in other representations.

VI. Conclusion and discussion

In this paper we showed that the proper translational symmetry for the problem of an electron in a constant magnetic field should be described by the magnetic translational group. The phases, in this description, fit into their proper place, so that translation invariance is exhibited as transformation of energy eigenstates within the same Landau level. We believe that similar discussions can be applied to the study of monopole harmonics [16] which are just wavefunctions of the Landau problem on a sphere [17], although the symmetry operators are the rotational operators which have more complicated commutation relations. On the whole our discussion here is quite straightforward. A more sophisticated discussion of the phase or the cocycle as a geometric object of the path integral can lead to the concept of anomaly breaking as in Manton [8].

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