Low-Temperature Series Expansions for the Ising Model on a Checkerboard Lattice with First and Second Neighbour Interactions

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We have calculated the low-temperature series expansions of the spontaneous magnetization and susceptibility of the Ising model on a checkerboard lattice with first and second neighbour interactions to the 22nd and 19th order respectively. We use the Padé approximants to estimate the critical exponents. Our results are consistent with the universality hypothesis which predicts that all two-dimensional Ising models have the same critical exponents $\beta = 0.125$ and $\gamma' = 1.75$.

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I. Introduction

The Ising model has a long history [1, 2]. In 1920, Lenz proposed a model of ferromagnetism such that, if an interaction is introduced between nearest-neighbouring spins, then at sufficiently low temperatures, the spins would be aligned. His student, Ising, solved the model in one dimension only [3]. The partition function of the square-lattice Ising model in the absence of a magnetic field was derived by Onsager [4]. The spontaneous magnetization of this model was also obtained by Onsager, but he never published his derivation. Yang was the first to publish a detailed derivation of the spontaneous magnetization, giving the critical exponent β as 0.125 [5]. The zero-field susceptibility of the square-lattice Ising model is still unsolved, but the critical exponents $\gamma(T > T_c)$ and $\gamma'(T < T_c)$ are known to be 1.75 [6]. The square-lattice Ising models with interactions beyond first neighbours are unsolvable by existing methods. According to the universality hypothesis, the critical exponents are expected to be the same for all two-dimensional Ising models.

The square-lattice Ising model, with first and second neighbour interactions, was first studied by Dalton and Wood [7]. They derived the low-temperature series expansions of the spontaneous magnetization and the zero-field susceptibility up to the 17th and 13th orders, respectively. Lee and Lin [8] extended their series to the 24th and 20th order, respectively. Recently Lin and Kao [9] developed a new method to calculate the lattice constants and obtained two more terms in the series expansions.

The Ising model on a checkerboard lattice with first and second neighbour interactions, as shown in Fig. 1, was first considered by Lin [10], who calculated the low-temperature series expansions for the spontaneous magnetization up to the 9th order. Lin and Liu [11] extended this

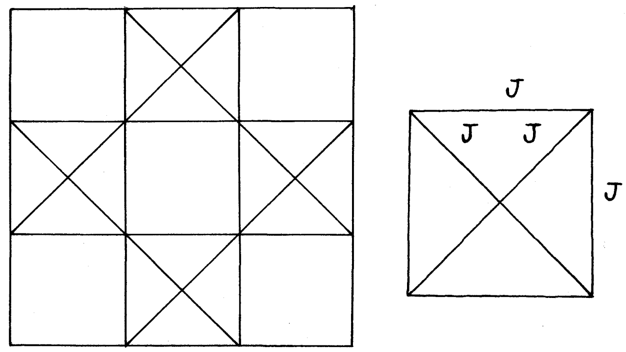

FIG. 1. The checkerboard lattice.

result to the 12th order. Recently Lin *et al.* [12] calculated the low-temperature series expansions of the spontaneous magnetization and the zero-field susceptibility of this model to the 16th and 13th order, respectively. In this paper, we extend both series expansions to six more terms. We shall describe the model and the series expansions in section II. In section III, we use the Padé approximants to estimate the critical temperature and the critical exponents. A conclusion is given in section IV.

II. Series expansions

We consider the Ising model on a checkerboard lattice with coupling constant J as shown in Fig. 1. The Hamiltonian is

$$H = -J \sum_{<ij>} \sigma_i \sigma_j ; \qquad (1)$$

where σ_i is a dummy variable which can take the values ± 1 and the summations are over nearest neighbour pairs and next-nearest neighbour pairs.

The method of low-temperature series expansions is well known [2]. The spontaneous magnetization $M(x)$ and the zero-field susceptibility $\chi(x)$ are expanded in a power series in x where

$$x = \exp(-4J/kT) . \qquad (2)$$

The series expansions are

$$M(x) = 1 - 2 \sum_{r=1}^{\infty} r g_r(x) ; \qquad (3)$$

$$\chi(x) = \sum_{r=1}^{\infty} r^2 g_r(x) = x^3 + \cdots ; \qquad (4)$$

where

$$g_r(x) = \sum_{n \le 3r} g(n;r)x^n. \tag{5}$$

The lattice constant $g(n;r)$ is associated with a set of graphs where each graph (which may consists of several disjoint diagrams) has r vertices and $b = 3r - n$ bonds [13]. The reduced zero-field susceptibility is defined by

$$\chi_0(x) = \chi(x)/x^3 = 1 + \ldots. \tag{6}$$

We have calculated the low-temperature series expansions of the spontaneous magnetization up to the 22th order and the reduced zero-field susceptibility up to the 20th order. The results are:

$$\begin{aligned} M(x) = 1 &- 2x^3 - 12x^5 - 2x^6 - 54x^7 - 28x^8 - 268x^9 - 372x^{10} - 1418x^{11} \\ &- 3006x^{12} - 9320x^{13} - 22414x^{14} - 65362x^{15} - 165040x^{16} \\ &- 475540x^{17} - 1303572x^{18} - 3327022x^{19} - 9517038x^{20} \\ &- 28781970x^{21} - 77163282x^{22} - \ldots, \end{aligned} \tag{7}$$

$$\begin{aligned} \chi_0(x) = 1 &+ 12x^2 + 12x^3 + 81x^4 + 176x^5 + 620x^6 + 2184x^7 + 5415x^8 \\ &+ 20108x^9 + 55668x^{10} + 179508x^{11} + 543369x^{12} + 1632682x^{13} \\ &+ 5053960x^{14} + 15726128x^{15} + 45247339x^{16} + 140409905x^{17} \\ &+ 465781913x^{18} + 1696378377x^{19} + \ldots. \end{aligned} \tag{8}$$

The lattice constants $c(n;r)$ of connected graphs for $n \le 22$ are listed in Tables I and II. A connected graph has the special property that a connected path exists for two arbitrary vertices.

III. Critical exponents

The Padé approximants were first applied to critical phenomena by Baker in 1961 [14]. The $[m;n]$ Padé approximant to a function $F(x)$ is the ratio of a polynomial $N(x)$ of degree m to another polynomial $D(x)$ of degree n such that the series expansion of $N(x)/D(x)$ agrees with the series expansion of $F(x)$ through order $m + n$. The critical point x_c and the critical exponent of a function $G(x)$ can be estimated from the poles and residues of the Padé approximants to the $(d/dx)\log G(x)$ series [14]. The Padé approximants of sufficiently high order will exactly represent rational functions, whose only singularities are finitely many poles. For two-dimensional Ising models with nearest-neighbor interactions, the spontaneous magnetization can be derived exactly and the logarithmic derivative of the functions representing spontaneous magnetization are rational functions. Therefore the method of Padé approximant is considered as the best method to estimate the critical point and exponent for spontaneous magnetization. For irrational functions, the Padé approximant can never represent exactly the original function. Nevertheless, this method can still be used to get a good approximation [14]. The estimated values of critical point and exponent will slowly converge to the exact values.

TABLE I. Lattice constants for n · 19:

n,r	c(n,r)	n,r	c(n,r)	n,r	c(n,r)	n,r	c(n,r)
5,2	3	6,3	2	6,4	0.5	7,3	9
8,4	18.5	8,5	6	9,4	25	9,5	13
9,6	8	9,7	1	10,5	96	10,6	51
10,7	12	10,8	3	11,5	71	11,6	123
11,7	113	11,8	41	11,9	13	11,10	2
12,6	437	12,7	320	12,8	172	12,9	84
12,10	29	12,11	8	12,12	0.5	13,6	195
13,7	913	13,8	962	13,9	538	13,10	290
13,11	118	13,12	38	13,13	6	14,7	1754
14,8	2017.5	14,9	1668	14,10	1133	14,11	686
14,12	317.5	14,13	158	14,14	46	14,15	6
15,7	543	15,8	5429	15,9	6756	15,10	4680
15,11	3130	15,12	1852	15,13	1061	15,14	526
15,15	213	15,16	60	15,17	9	15,18	2
16,8	6629.5	16,9	12620	16,10	14722	16,11	11860
16,12	8587	16,13	5724	16,14	3535	16,15	1946
16,16	919.5	16,17	356	16,18	98	16,19	28
16,20	3	17,8	1479	17,9	28317	17,10	41879
17,11	37049	17,12	29430	17,13	21214	17,14	14855
17,15	9850	17,16	6068	17,17	3334	17,18	1580
17,19	626	17,20	234	17,21	55	17,22	8
17,23	1	18,9	23660	18,10	76482	18,11	111186
18,12	109545.5	18,13	85052	18,14	63993	18,15	45944
18,16	31337	18,17	20116	18,18	12007	18,19	6500
18,20	3285.5	18,21	1558	18,22	574	18,23	162
18,24	35.5	18,25	4	19,9	4067	19,10	133355
19,11	246931	19,12	276586	19,13	260120	19,14	216830
19,15	171568	19,16	129704	19,17	94049	19,18	64346
19,19	41071	19,20	24326	19,21	13984	19,22	7638
19,23	3629	19,24	1462	19,25	483	19,26	116
19,27	18	19,28	2				

TABLE II. Lattice constants for $n = 20, 21, 22.$

n,r	c (n,r)	n,r	c (n,r)	n,r	c (n,r)	n,r	c (n,r)
20,10	81670	20,11	434686	20,12	273880.5	20,13	872230
20,14	740098	20,15	618952	20,16	494444.5	20,17	379041
20,18	280825	20,19	199981	20,20	105797	20,21	87124
20,22	54829	20,23	32696	20,24	17987.5	20,25	9054
20,26	3924	20,27	1520	20,28	472	20,29	124
20,30	24.5	20,31	4	20,32	0.5	21,10	11025
21,11	584568	21,12	1411304	21,13	2008255	21,14	2142967
21,15	2003248	21,16	1738768	21,17	1504402	21,18	1201170
21,19	923386	21,20	681376	21,21	481599	21,22	330474
21,23	234732	21,24	149882	21,25	89991	21,26	50515
21,27	25576	21,28	13100	21,29	6350	21,30	3346
21,31	2053	21,32	1412	21,33	918	21,34	490
21,35	205	21,36	71	21,37	25	21,38	9.5
21,39	5.5	21,40	0.5	22,11	272790	22,12	2297564
22,13	4724472	22,14	6394470	22,15	5899367	22,16	5524464
22,17	5148602	22,18	4216280	22,19	3442356	22,20	2732642
22,21	2050112	22,22	1491028	22,23	1126494	22,24	781070
22,25	556126	22,26	369708	22,27	235084	22,28	137398
22,29	84184	22,30	53695	22,31	37442	22,32	28374
22,33	22258	22,34	16360	22,35	10486	22,36	5753
22,37	2876	22,38	1469	22,39	785	22,40	394
22,41	164	22,42	53	22,43	12	22,44	0.5

TABLE III. Estimation of the critical point x_c from the poles of the $[n + j, n]$ Padé approximants to the $(d/dx) \log M(x)$ series.

n	$j = -1$	$j = 0$	$j = +1$
5	0.3397	0.3460	0.3476
6	0.3485	0.3456	0.3442
7	0.3455	0.3427	0.3431
8	0.3432	0.3432	0.3431
9	0.3431	0.3427	0.3440

TABLE IV. Estimation of the critical point x_c from the poles of the $[n+j; n]$ Padé approximants to the $(d/dx)\log\hat{\chi}_0(x)$ series.

n	$j=-1$	$j=0$	$j=+1$
5	0.3231	0.3255	0.3346
6	0.3217	0.3341	0.3345
7	0.3383	0.3352	0.3348
8	0.3337	0.3348	0.3349
9	0.3354		

TABLE V. Estimation of the critical exponent β from the Padé approximants to the $(0.342-x)(d/dx)\log M(x)$ series at $x_c = 0.342$.

n	$j=-1$	$j=0$	$j=+1$
6		0.1308	0.1256
7	0.1282	0.1253	0.1258
8	0.1244	0.1430	0.1256
9	0.1241	0.1246	0.1244

Consider first the critical point x_c. The spontaneous magnetization and the susceptibility should have the same critical point. The estimated critical points from the Padé approximants are shown in Tables III and IV for $M(x)$ and $\hat{\chi}_0(x)$ respectively. Since the series for $\hat{\chi}_0$ is shorter than the series for M, the corresponding estimated values are far from the exact value. From the estimated values of these two Tables, we conclude that

$$x_c = 0.342 \pm 0.01. \tag{9}$$

Consider next the critical exponent β. In general, the estimated values of the critical point converge faster than the corresponding values of the critical exponent [14]. If a good estimate of the critical point is known, a biased estimate of the critical exponent is obtained by forming the Padé approximants to the series $(x_c - x)(d/dx)\log M(x)$ and evaluating them at $x = x_c$ [14]. We made a biased estimate of β by assuming $x_c = 0.342$ and the result is shown in Table V. From Table V, we obtain

$$\beta = 0.125 \pm 0.005. \tag{10}$$

Similarly, we made a biased estimate of the γ' result as shown in Table VI. From Table IV, we obtain

$$\gamma' = 1.75 \pm 0.10. \tag{11}$$

TABLE VI. Estimation of the critical exponent γ' from the Padé approximants to the $(0.342 - x)(d/dx)\log\chi_0(x)$ series at $x_c = 0.342$.

n	$j = -1$	$j = 0$	$j = +1$
6	2.0740	1.6862	1.7222
7	1.7085	1.6805	1.5076
8	1.8475	1.7566	2.3132
9	1.8160		

IV. Conclusion

The spontaneous magnetization and the zero-field susceptibility of the Ising model on a checkerboard lattice with first and second neighbour interactions are studied by the method of series expansions and Padé approximants. We use the Padé approximants to estimate the critical point and the critical exponents. The results are given in Eqs. (9-11). Our results are consistent with the universality hypothesis which predicts that all two-dimensional Ising models have the same exponents $\beta = 0.125$ and $\gamma' = 1.75$.

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