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Generalized Coulomb Potential and Field Equations in Accelerated Frames Based on the Wu Transformation

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A Modified Poisson equation and the Coulomb potential of a point charge in a constant-linear-acceleration frame are obtained. The generalized Maxwell equations for both inertial and linearly accelerated frames are discussed. These results are obtained on the basis of the 'Wu transformation', which is a generalized Lorentz transformation for constant-linear-acceleration frames and which reduces to the Lorentz transformation with four-dimensional symmetry in the limit of zero acceleration. Some physical implications of the results regarding the Schroedinger and Dirac equations are discussed.

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I. Introduction

Within the framework of classical mechanics, we have satisfactory Galilean transformations for inertial frames and their extension to constant-linear-acceleration (CLA) frames. Galilean transformations for inertial frames are known to be only approximately true and have been satisfactorily improved in modern physics by the use of the Lorentz transformations. Nevertheless, the corresponding extension of the Lorentz transformations to CLA frames has not been developed satisfactorily. The first attempt to generalize the invariant laws of electromagnetism to accelerated frames of reference was made by Einstein himself in 1907 [1]. Unfortunately, Einstein's transformations for accelerated frames do not reduce to the four-dimensional Lorentz transformations in the limit of zero acceleration. In other words, they do not satisfy the 'principle of limiting four-dimensional symmetry' which requires that any accelerated transformation must reduce to a transformation with the four-dimensional symmetry of the Lorentz group in the limit of zero acceleration. Therefore, they are not completely satisfactory.

Many years later, in 1943, Møller obtained a transformation for an inertial frame F_I and a uniformly accelerated frame F, moving along the x-axis [2]. Møller used Einstein's field equation for vacuum and assumed that $g_{11} = g_{22} = g_{33} = -1$ and that g_{00} is time independent. He obtained the line element

$$ds^2 = \left(1 + \frac{g}{c^2} x\right)^2 c^2dt^2 - dx^2 - dy^2 - dz^2; \tag{1}$$

and the accelerated transformations

$$
\begin{aligned}
X &= \frac{c^2}{g}\left(\cosh\frac{gt}{c} - 1\right) + x\cosh\left(\frac{gt}{c}\right);\\
Y &= y;\\
Z &= z;\\
cT &= \frac{c^2}{g}\sinh\left(\frac{gt}{c}\right) + x\sinh\left(\frac{gt}{c}\right);
\end{aligned}
\tag{2}
$$

where g is a constant acceleration. Note that cT, X, Y, Z denote the coordinates of an event in an inertial frame F_I and ct, x, y, z denote the coordinates of the same event in the accelerated frame F.

In 1972, Ta-You Wu and Y. C. Lee derived the same transformation (2) based on a purely 'kinematic approach', without using Einstein's field equations at all [3]. Instead, they made two assumptions that the invariant spacetime interval in the CLA frame F moving along the x-axis is given by $ds^2 = g_{00}(x)c^2dt^2 - dx^2 - dy^2 - dz^2 = c^2dT^2 - dX^2 - dY^2 - dZ^2$ and that the local Lorentz contraction holds, i.e. $\partial X/\partial x|_T = \sqrt{1 - v^2(x,T)/c^2}$.

The problem with the accelerated transformations discussed by Møller, Wu, Lee and others is that they do not reduce to the Lorentz transformations in the limit of zero-acceleration [3].

II. The Wu transformation

In 1997, Hsu and Hsu investigated a new 'kinematic approach' based on the 'principle of limiting four-dimensional symmetry.' They derived an interesting coordinate transformation (called the Wu transformation) [4] between an inertial frame F_I and a constant-linear-acceleration (CLA) frame F. Their resultant transformation can be extended by including constant translations (w_o, x_o, y_o, z_o),

$$
\begin{aligned}
w_I &= \gamma\beta(x + 1/\alpha\gamma_o^2) - \beta_o/\alpha\gamma_o + w_o;\\
x_I &= \gamma(x + 1/\alpha\gamma_o^2) - 1/\alpha\gamma_o + x_o;\\
y_I &= y + y_o;\\
z_I &= z + z_o;
\end{aligned}
\tag{3}
$$

where $w_I, x_I, y_I,$ and z_I denote the coordinates of an event in the inertial frame F_I, while w, x, y and z denote the coordinates of the same event in the CLA frame F. For the operational meaning of the 'time' w, we stress that it is a variable measured in the unit of length (e.g., cm) and that it can be physically realized by a set of 'computerized clocks' in the CLA frame F [4, 5]. In general, the 'time' w has nothing to do with the constant speed of light c. The CLA frame F moves with a time-dependent velocity $\beta = \alpha w + \beta_o$ along the x_I direction. One can verify that the inhomogeneous Wu transformation (3) displays the 4-dimensional symmetry of the Poincaré group in the limit of zero acceleration α: $w_I = \gamma_o(w + \beta_o x) + w_o, x_I = \gamma_o(x + \beta_o w) + x_o, y_I = y + y_o, z_I = z + z_o$. We know that when $w_o = x_o = y_o = z_o = 0$, the Poincaré group reduces to the Lorentz group. In an inertial frame, if one wishes, one may set $w_I = ct_I$, $w = ct$, where c denotes the velocity of light as measured in the inertial Frame F_I. It is important to remark that β and γ depend - unlike in special relativity - on both the initial velocity v_0 (or β_0) and the constant acceleration α. We have the following relations:

$$\begin{aligned} \beta &= \alpha w + \beta_o; \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-(\alpha w+\beta_o)^2}}; \\ \beta_o &= v_0/c; \qquad \gamma_o = \frac{1}{\sqrt{1-\beta_o^2}}. \end{aligned} \tag{4}$$

The operational meaning of the "constant acceleration" α in (3) is as follows: If a particle is at rest in a CLA frame F, then the change of the particle's energy per unit length is a constant, as measured from an inertial frame [4]. This is precisely what happens to charged particles in a linear accelerator.

The Wu transformation (with zero constant translations) reduces, unlike the transformation of Møller, to the Lorentz transformation in the limit of zero acceleration. It also reduces to the Galilean transformation for accelerated frames for small velocities. It can also be shown that the set of Wu transformations for CLA frames form a group, which is called the 'Wu group'. The group property of the Wu transformations is shown in Ref. [4]. The Wu group is the simplest generalization of the Lorentz and Poincaré groups (provided the usual relation $\beta = \alpha w + \beta_o$ holds) and implies the Lorentz and Poincaré groups as special limiting cases when the acceleration α approaches zero. Thus, the Wu transformation describes the physical properties of spacetime of both inertial frames and constant-linear-acceleration frames.

From the Wu transformation (3) one obtains the following metric tensor $g_{\mu\nu}$:

$$g_{\mu\nu} = \begin{pmatrix} \gamma^4(1/\gamma_o^2+\alpha x)^2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{5}$$

Note that it does not reduce to the Møller metric tensor in (1) when β_o approaches 0. The reason for this is that the physical times in the Møller transformation (2) and the Wu transformation (3) for the CLA frames are quite different (i.e., different transformation properties.) Nevertheless, the metric tensor (5) reduces to the Minkowski metric tensor in the limit of zero acceleration. It is interesting to note that the spacetime of CLA frames characterized by the metric tensor (5) is flat because the Riemann curvature tensor $R_{\mu\nu\rho\lambda}$ calculated from $g_{\mu\nu}$ in (5) vanishes. This property is related to the existence of the Wu transformation (3) for all spacetime. The curvature of the spacetime of F_I is not changed by the change of variables in (3), so that the curvature of the spacetime of a CLA frame F must be the same as that of the inertial frame F_I.

III. Generalized wave equations for accelerated frames

The 4-dimensional wave equation for a scalar field ϕ in an inertial frame has the form

$$(\partial_\mu \partial^\mu)_I \phi = \left[\frac{\partial^2}{\partial w_I^2} - \frac{\partial^2}{\partial x_I^2} - \frac{\partial^2}{\partial y_I^2} - \frac{\partial^2}{\partial z_I^2}\right] \phi = 0. \tag{6}$$

If we express the d'Alembertian operator in (7) in terms of the coordinates in the accelerated frame F, we obtain the generalized 4-dimensional wave equation for a scalar field in a CLA frame:

$$(D_\mu D^\mu)_I \phi = \left[\frac{1}{W^2}\frac{\partial^2}{\partial w^2} - \nabla^2 - 2\alpha\beta\gamma^2\frac{1}{W^2}\frac{\partial}{\partial w} - \alpha\gamma^2\frac{1}{W}\frac{\partial}{\partial x}\right]\phi = 0, \tag{7}$$

where D_μ denotes the partial covariant derivative associated with $g_{\mu\nu}$ in (5), and

$$W = W(w,x) = \sqrt{g_{00}(w,x)} = \gamma^2(1/\gamma_o^2 + \alpha x) > 0. \tag{8}$$

Let us consider a scalar field ϕ which does not depend on the time:

$$\frac{\partial\phi}{\partial w} = 0. \tag{9}$$

All derivatives of ϕ with respect to w vanish. In this case, the transformed Laplacian operator ∇_I^2 can be obtained from (7) and (8):

$$\nabla_I^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \alpha\gamma^2\frac{1}{W}\frac{\partial}{\partial x}. \tag{10}$$

Although γ and W in (10) involve time, however, if we plug W from (9) into (10), we see that the transformed Laplacian is time-independent. Now we can obtain the exact form of the generalized Poisson equation for a point source at the origin of a CLA frame:

$$\left(\nabla^2 + \frac{\alpha\gamma_o^2}{1+\alpha\gamma_o^2 x}\frac{\partial}{\partial x}\right)\phi(x,y,z) = -Q\delta(x,y,z), \tag{11}$$

where Q is the numerical constant that characterizes the strength of the coupling of the scalar field to the source, and $\delta(x,y,z)$ is the usual Dirac delta function which is zero everywhere except at the origin of the CLA frame.

It appears that there is no general and exact solution for this kind of partial differential equation in the literature. Therefore we have to restrict ourselves to an approximate solution for the potential ϕ. To obtain the approximate solution, we have to expand the Laplacian operator for small values of α. Up to the second order in α we have

$$\nabla_I^2 = \nabla^2 + \frac{\alpha\gamma_o^2}{1+\alpha\gamma_o^2 x}\frac{\partial}{\partial x} = \nabla^2 + \alpha\gamma_o^2\frac{\partial}{\partial x} - \alpha^2\gamma_o^4 x\frac{\partial}{\partial x} + \cdots. \tag{12}$$

This yields the following approximation for the generalized Poisson equation for a scalar field in a CLA frame:

$$\left(\nabla^2 + \alpha\gamma_o^2\frac{\partial}{\partial x} - \alpha^2\gamma_o^4 x\frac{\partial}{\partial x} + \cdots\right)\phi(x,y,z) = -Q\delta(x,y,z). \tag{13}$$

The covariant Maxwell equations in CLA frames can be written in the familiar form

$$\begin{aligned} &g^{\mu\nu}D_\mu f_{\nu\alpha} = j_\alpha, \\ &\partial_\lambda f_{\mu\nu} + \partial_\mu f_{\nu\lambda} + \partial_\nu f_{\lambda\mu} = 0 \end{aligned} \tag{14}$$

where $f_{\mu\nu} = D_\mu a_\nu - D_\nu a_\mu = \partial_\mu a_\nu - \partial_\nu a_\mu$ and D_μ denotes the covariant derivative associated with the metric tensor $g_{\mu\nu}$ in (5).

The equation for the Coulomb potential is given by the zeroth component of equation (14):

$$-g^{\mu\nu} D_\mu f_{\nu 0} = \left(\nabla^2 - \frac{\alpha}{\gamma_0^2 + \alpha x} \partial_1 \right) a_0 = -j_0. \tag{15}$$

The difference of the sign for the term $(\alpha/[\gamma_0^2 + \alpha x])\partial_1 a_0$ in (15) and (12) is due to the different transformation properties of a scalar field ϕ and the zeroth component a_0 of a 4-vector field.

The following steps are purely mathematical. We have to solve the above equation for the potential $a_0(x, y, z)$ which depends on the spatial coordinates in the F frame. For simplicity, we introduce a new parameter $\alpha^* = \alpha\gamma_0^2$.

IV. The potential of a point charge in an accelerated frame

IV-1. First order approximation

Up to the first order in the acceleration, the modified Poisson equation (15) for a point charge at rest in a CLA frame has the form

$$\left(\nabla^2 - \alpha^* \frac{\partial}{\partial x} \right) a_0 = -q\delta(x, y, z). \tag{16}$$

We make the 'Ansatz'

$$a_0 = \frac{q}{4\pi r}(1 + \kappa x), \tag{17}$$

and solve for κ, keeping only terms up to the first order in α^*. We obtain, after a straightforward calculation, a modified Coulomb potential to the first order in α^*

$$a_0 = \frac{q}{4\pi r}\left(1 + \frac{\alpha^*}{2} x \right). \tag{18}$$

IV-2. Second order approximation

Now let us consider the modified Poisson equation up to the second order in α^*. The approximate modified Poisson equation has the form

$$\left(\nabla^2 - \alpha^* \frac{\partial}{\partial x} + \alpha^{*2} x \frac{\partial}{\partial x} \right) a_0 = -q\delta(x, y, z). \tag{19}$$

We make the 'Ansatz'

$$a_0 = \frac{q}{4\pi r}\left(1 + \frac{\alpha^*}{2} x + \lambda x^2 + \mu (y^2 + z^2) \right) \tag{20}$$

and solve for the parameters λ and μ, keeping only terms up to the second order in α^*.

The solution for the Coulomb potential up to the second order in α^* is found to be approximately

$$
\begin{aligned}
a_0 &= \frac{q}{4\pi r}\left(1 + \frac{\alpha^*}{2}x + \frac{\alpha^{*2}}{4}x^2 + \frac{3\alpha^{*2}}{8}(y^2 + z^2)\right) \\
&= \frac{q}{4\pi r}\left(1 + \frac{\alpha^*}{2}x + \frac{\alpha^{*2}}{8}(2x^2 + 3y^2 + 3z^2)\right).
\end{aligned}
\tag{21}
$$

We notice that this solution does not satisfy the boundary condition $\phi = 0$ for $r \to \infty$. Therefore we have to find a form for ϕ which is compatible with the boundary condition.

For small α^*, a possible form which satisfies the boundary condition $a_0 = 0$ at infinity could be

$$
a_0 = \frac{q}{4\pi r(1 - [\alpha^*/2][x + (3\alpha^*/4)(y^2 + z^2) + \cdots])}. \tag{22}
$$

We may remark here that a solution, which satisfies the boundary condition, could have a different form as well. However, they all have the same approximation (21) up to the second order in α^*.

V. The Schroedinger and the Dirac equations in CLA frames

Since we have a time-independent potential, we can consider the time-independent Schroedinger equation for a hydrogenlike atom in an accelerated frame. The Schroedinger equation is the "non-relativistic" (or "small momentum") approximation of the Klein-Gordon equation for a scalar field. We find that, for small x, y, z (atomic domain) and small acceleration α^*, it has the following approximate form (in the usual notations in a particular CLA frame)

$$
\begin{aligned}
&-\frac{\hbar^2}{2m}\left((1 + \alpha^* x)\,\nabla^2 + \alpha^*\frac{\partial}{\partial x}\right)\psi(x, y, z) \\
&-\frac{Ze^2}{r}\left(1 + \frac{\alpha^*}{2}x\right)\psi(x, y, z) = E\psi(x, y, z),
\end{aligned}
\tag{23}
$$

to the first order in α^*, where $\alpha^* = \alpha\gamma_0^2$ and $\alpha^* x$, $\alpha^* y$, $\alpha^* z$ are much smaller than 1.

Of course, for large r (compared with the Bohr radius $r_B \approx 10^{-8}$ cm), one has to use the modified Coulomb potential (22), which satisfies the boundary condition at $r \to \infty$. It seems to be rather impossible to find an analytic solution for this equation. Nevertheless, one can use perturbation theory to obtain corrections for the energy levels of a hydrogenlike atom in an accelerated frame up to the second order in α^*. The term involving $\alpha^*\frac{\partial}{\partial x}$ in (23) will lead to a very very small imaginary energy in perturbation theory. This is in harmony with classical electrodynamics, in which an electric charge with constant-linear-acceleration can emit a very small amount of radiation. Our calculations show that the corrections for energy levels in (23) are either zero or extremely small and negligible for possible accelerations on earth laboratories (currently available voltage gradients are about 70 MeV per meter for linear accelerators). Physically, this result suggests that the numerical values of energy levels are almost unchanged by constant-linear-accelerations.

In accelerated frames, the speed of light is in general not a universal constant. The constant c = 29979245800 cm/sec for the speed of light in inertial frames has neither operational meaning nor physical significance in CLA frames. Thus, we postulate an invariant action S for a classical charged particle interacting with a 4-potential a_μ in a CLA frame without involving the constant speed of light c:

$$S = \int (mds - \bar{e}a_\mu dx^\mu); \tag{24}$$

where $ds^2 = dx_\mu dx^\mu = g_{00}dw^2 - dr^2$ and $\bar{e}$ is the electric charge measured in the electromagnetic units [4]. By the variational calculus, it leads to the following canonical momentum P_μ, [6]

$$P_\mu = p_\mu - \bar{e}a_\mu; \tag{25}$$

where the particle's momentum p_μ is given by

$$p_\mu = m\frac{dx_\mu}{ds} = \frac{m}{\sqrt{g_{00} - \beta^2}}\left(1; -\frac{dx}{dw}; -\frac{dy}{dw}; -\frac{dz}{dw}\right); \tag{26}$$

$$dx_i = g_{ik}dx^k = (-dx; -dy; -dz); \qquad \beta^2 = (dx/dw)^2 + (dy/dw)^2 + (dz/dw)^2:$$

The expression for particle's momentum shows that the maximum speed (measured in terms of the 'time' w) of a physical particle is a spacetime-dependent function g_{00} rather than a constant in a CLA frame. Furthermore, if the function $g_{00} = W^2$ vanishes, the velocities of all physical particles vanish too. This can be seen from the velocity transformation based on the Wu transformation (3): $dx/dw = W[(1 - \beta dx_I/dw_I)/(dx_I/dw_I - \beta)]$. Strange phenomena occurs at $x = -1/(\alpha\gamma_0^2)$, which is a 'wall singularity'. When the acceleration α approaches zero, the 'singularity wall' in a CLA frame moves to infinity. For a finite α, W becomes infinity as x approaches $-1/(\alpha r_0^2)$, because γ also goes to infinity.

Since $p_\mu p^\mu = m^2$; as one can see from the above expression for p_μ, we have the invariant relation for the canonical momentum P_μ

$$(P_\mu + \bar{e}a_\mu)(P^\mu + \bar{e}a^\mu) = m^2: \tag{27}$$

This suggests that the generalized Klein-Gordon equation for a charged scalar particle in the electromagnetic 4-potential a_μ takes the form

$$\left[(iJD_\mu + \bar{e}a_\mu)(iJD^\mu + \bar{e}a^\mu) - m^2\right]\phi = 0; \tag{28}$$

where D_μ denotes the covariant derivative associated with the metric tensor $g_{\mu\nu}$:

For the Dirac equation in CLA frames, one must be aware of the spacetime dependence of the metric tensor $g_{\mu\nu}$ in CLA frames. In contrast to the constant Minkowski metric tensor in inertial frames, this property implies a spacetime dependence of the Dirac matrices because of the anti-commutation relation

$$\{\Gamma_\mu; \Gamma_\nu\} = 2g_{\mu\nu} \tag{29}$$

Thus, one must postulate a symmetrized action in order to preserve the same symmetry of particle and anti-particle in inertial frames. Such a symmetrization was not carried out in the second paper in ref. 4. The invariant action S_ψ for a free electron field ψ in CLA frames is assumed to be

$$S_\psi = \int L_\psi d^4x; \tag{30}$$

$$L_\psi = \frac{1}{2}\sqrt{-g}\,\bar{\psi}\Gamma^\mu i\partial_\mu\psi - \frac{1}{2}\sqrt{-g}\,(i\partial_\mu\bar{\psi})\Gamma^\mu\partial_\mu\psi - m\sqrt{-g}\,\bar{\psi}\psi;$$

$$\sqrt{-g} = W; \qquad \Gamma^\mu = (\gamma_D^0/W, \gamma_D^1, \gamma_D^2, \gamma_D^3);$$

where $\gamma_D^\mu = (\gamma_D^0, \gamma_D^1, \gamma_D^2, \gamma_D^3)$ are the usual constant Dirac matrices. Using integration by parts in the action, the Lagrangian L_ψ can be written as

$$L_\psi = W\bar{\psi}\Gamma^\mu i\nabla_\mu\psi - mW\bar{\psi}\psi; \tag{31}$$

$$\nabla_\mu = \left(\partial_0 + \frac{1}{2}(\partial_k W)\gamma_D^0\gamma_D^k, \partial_1, \partial_2, \partial_3\right); \quad W = \sqrt{-g}; \quad g = \det g_{\mu\nu}.$$

The Lagrangian L_ψ leads to generalized Dirac equations for both inertial and CLA frames: [8]

$$(iJ\Gamma^\mu\nabla_\mu - m)\psi = 0; \quad (\nabla_\mu\bar{\psi})(-iJ\Gamma^\mu) - m\bar{\psi} = 0; \tag{32}$$

where J is a constant. (see Eq. (34) below.)

VI. Remarks

We may remark that the Møller transformation (2) for accelerated frames can be generalized so that it reduces to the Lorentz transformation in the limit of zero acceleration [7].

To the second order in $\alpha^* = \alpha\gamma_o^2$, the approximate solution for the static scalar field to equation (11) is given by

$$\phi = \frac{Q}{4\pi r}\left(1 - \frac{\alpha^*}{2}x + \frac{\alpha^{*2}}{8}(2x^2 - y^2 - z^2)\right); \tag{33}$$

where $\alpha^* = \alpha\gamma_o^2$. The scalar equation (11) resembles the static Yukawa equation for the pion, which corresponds to a pseudo-scalar field.

In equation (23), the invariant charge $\tilde{e}$ in CLA frames corresponds to the usual e/c, where e is measured in electric static units. Also, the potential a_μ corresponds to A_μ/c, where A_μ is the usual 4-potential.

In CLA frames, both the speed of light and the Planck constant are no longer universal constants [4, 8]. The energy-momentum four-vector p^μ is related to the wave four-vector k^μ by the relation

$$p^\mu = Jk^\mu \tag{34}$$

where $J = 3.5177293 \times 10^{-38}$ g·cm. In inertial frames p^0 corresponds to the usual expression E/c^2 and k^0 corresponds to ω/c, where c denotes the speed of light.

In special relativity, c has the value 29979245800 cm/sec in all inertial frames [9]. As we go into CLA frames, the speed of light is no longer a universal constant. However, this does not hinder us from the formulation of a spacetime theory for CLA frames because we use (w_I, x_I, y_I, z_I) and (w, x, y, z) as variables in all equations. Furthermore, if one consistently uses (w_I, x_I, y_I, z_I) and (w, x, y, z) for inertial and linearly accelerated frames, one finds that the truly universal constants for physics in both inertial and noninertial frames are the electromagnetic coupling strength $\alpha_e = 1/137.03604 = e^2/(4\pi J)$ and the quantum constant $J = 3.5177293 \times 10^{-38}$ g·cm [5, 8], rather than the usual constants e (in electrostatic units), $\hbar$ and $c = 29979245800$ cm/sec within the framework of special relativity.

Acknowledgments

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