

Two-Oscillator Representation of the Landau Problem

M. K. Fung and Y. F. Wang

*Department of Physics, National Taiwan Normal University,
Taipei, Taiwan 116, R.O.C.*

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We have quantised the two oscillation modes of the Landau problem. Coherent state techniques enable us to obtain in an easy way the path integral of the Landau problem. The Landau problem in a symmetric quadratic external potential is investigated along similar lines.

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I. Introduction

The quantum problem of an electron in a constant magnetic field is known as the Landau problem. No wonder, it is a fundamental problem of physics. Landau [1] solved the problem in the famous Landau gauge and he obtained wavefunctions which are plane wave solutions in one direction and harmonic oscillator solutions in the other. The energy levels are equally spaced like the harmonic oscillator but each level is highly degenerate. They became known as the Landau levels. Laughlin [2] re-examined the problem and solved it in the symmetric gauge. He obtained wavefunctions which respect rotational symmetry. These wavefunctions are crucial for the theory of quantum Hall effects. All of the above discussions focus on the calculation of the wavefunctions. A gauge choice must be adopted in each case. The Landau problem looks like an oscillation problem. Hence it is instructive to cast the problem in terms of creation and annihilation operators. The energy creation and annihilation operators were noticed long ago by Landau. However, the magnetic translation creation and annihilation operators have received little attention. The magnetic translation operators were discussed by Zak [3] and Brown [4] in the context of discussing translational symmetry in the presence of a magnetic field. The magnetic translational operators do not commute and are *oscillator-like*. They do commute with the velocity operators in a magnetic field. So it is easy to define the creation and annihilation operators for the magnetic translation as [5]. It is remarkable that these two sets of creation and annihilation operators can be defined in a gauge independent way. The magnetic translation creation and annihilation operators commute with the Hamiltonian and they account for the existence of the degeneracy. Recently Wiegmann and Zabrodin [6] found a deep relation between the magnetic translation group and the quantum group $U_q(sl_2)$ in the Hofstadter problem, a lattice version of the Landau problem. In [7] this quantum group symmetry was employed to discuss the degeneracy of the Landau problem with periodical boundary conditions. This paper attempts to show that the representation of the Landau problem in terms of two commuting sets of creation and annihilation operators can yield another insight into the problem.

This paper is organised as follows. In section 2 we review the classical theory of an electron in a constant magnetic field. Specifically we write down the equations of motion in terms of complex co-ordinates as a linear differential equation with constant coefficients. In this way we are in a better position to see how the problem can be extended and still be soluble. In section 3 we study the quantum problem and introduce the two commuting sets of creation and annihilation operators. Using the oscillator analogue we define the coherent state wavefunctions and then obtain in a simple way the path integral for the Landau problem. In section 4 we study the Landau problem in a symmetric quadratic potential. The problem is solved by means of the method of creation and annihilation operators in a gauge independent way. The wavefunctions in the symmetric gauge are then exhibited. Finally in section 5 we make our conclusion.

II. The solution for the classical problem

We consider the classical problem of an electron of charge e and mass m moving in a constant magnetic field \mathbf{B} . The equations of motion are given by the Lorentz force law

$$m\ddot{\mathbf{r}} = e\dot{\mathbf{r}} \times \mathbf{B}, \quad (1)$$

where we have adopted the convention $\hbar = c = 1$. The motion is essentially two dimensional. We can take the magnetic field \mathbf{B} along the z -axis, and the equations of motion for the x, y co-ordinates are given by

$$\ddot{x} = -\omega\dot{y}, \quad (2)$$

$$\ddot{y} = \omega\dot{x}, \quad (3)$$

where $\omega = |e|B/m$ is the cyclotron frequency. This set of equations is easily solved. However, we would prefer to cast the equations into a form like the problem of oscillations so that we can solve more general problem of the same type. Following Laughlin [2] we adopt the complex co-ordinates

$$z = x - iy. \quad (4)$$

In terms of the complex co-ordinates the equations of motion become

$$\ddot{z} + i\omega\dot{z} = 0, \quad (5)$$

which is a linear differential equation with constant coefficients. This is a typical oscillator problem. The solution is

$$z = z_0 + w_0 e^{-i\omega t}, \quad (6)$$

where $z_0 = x_0 - iy_0$ denotes the co-ordinates of the centre of the circular orbit and $|w_0|$ denotes the radius of the circular motion. The electron simply circulates anti-clockwisely around the orbit centre. The zero mode of oscillation is present. So this accounts for the huge degeneracy in the quantum problem.

In this problem we can add a constant electric field and the equation becomes a forced oscillator problem and is soluble. The orbit centre just drifts in a direction perpendicular to the electric field and this is well-known in plasma physics.

We can also add a rotationally invariant quadratic potential $\frac{1}{2}\epsilon m\omega^2(x^2 + y^2)$ to the problem and the equation is still soluble in the form

$$\ddot{z} + i\omega\dot{z} + \epsilon\omega^2 z = 0. \quad (7)$$

The solution is

$$z = a_0 e^{i\omega_1 t} + b_0 e^{-i\omega_2 t}, \quad (8)$$

where

$$\omega_{1,2} = \frac{\sqrt{1 + 4\epsilon} \mp 1}{2}\omega. \quad (9)$$

It is interesting to note that the two frequencies are in opposite sense. We expect that this problem is also exactly soluble in the quantum case.

III. The quantum problem

The quantum problem of an electron moving in a constant magnetic field had been solved long ago by Landau [1]. Landau worked out the wavefunctions in the famous Landau gauge. Fifty odd years later Laughlin [2] solved the problem in the symmetric gauge. Indeed, Landau's wavefunctions and Laughlin's wavefunctions are different representations of the same problem. Laughlin's wavefunctions respect rotational symmetry and are eigenstates of the angular momentum. Landau's wavefunctions are a kind of coherent state representation [8]. In this section we shall present a solution of the Landau problem in terms of creation and annihilation operators in a gauge-independent way. There are, in fact, for completeness two sets of creation and annihilation operators. This is not surprising since the classical equations of motion, better viewed in the complex co-ordinates, are a linear oscillation problem with two independent frequencies. The first set of creation and annihilation operators was first noted by Landau long ago. He remarked that in the Hamiltonian

$$H = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2), \quad (10)$$

the velocity operators \dot{x} and \dot{y} have a nonvanishing commutation relation

$$[\dot{x}, \dot{y}] = \frac{ieB}{m^2}, \quad (11)$$

because quantum mechanically,

$$\dot{x} = p_x - eA_x, \quad (12)$$

$$\dot{y} = p_y - eA_y, \quad (13)$$

where p_x and p_y are the conjugate momenta of the Landau problem. This is similar to the commutator of the one-dimensional oscillator for the co-ordinate and the conjugate momentum. So one can define

$$b = \sqrt{\frac{m}{2\omega}} (\dot{y} + i\dot{x}) \quad (14)$$

$$b^+ = \sqrt{\frac{m}{2\omega}} (\dot{y} - i\dot{x}), \quad (15)$$

where $\omega = |e|B/m$ is the cyclotron frequency. The creation operator b^+ and annihilation operator b obey the commutation relation

$$[b, b^+] = 1. \quad (16)$$

Hence the Hamiltonian can be written as

$$H = \left(b^+ b + \frac{1}{2} \right) \omega. \quad (17)$$

Besides, b^+ is a raising operator and b a lowering operator for the angular momentum J ,

$$[J, b^+] = b^+, \quad (18)$$

$$[J, b] = -b, \quad (19)$$

and the angular momentum operator J is found to be [5]

$$J = m(xy - yx) + \frac{eB}{2} (x^2 + y^2), \quad (20)$$

containing a mechanical part as well as a field part.

However, there are zero modes because of the translational invariance of the problem. This accounts for the degeneracy of states. Translational invariance is a little bit subtle in this problem because of the effects of the magnetic field. In the classical solution we see that the centre of the circular orbit can be located anywhere on the plane. In the quantum case the symmetry generators for translational invariance in the presence of a magnetic field were first discussed by Zak [3] and Brown [4], and can be written in a gauge independent form as

$$k_x = m\dot{x} - eBy, \quad (21)$$

$$k_y = m\dot{y} + eBx. \quad (22)$$

These are called the magnetic translation operators. Classically they correspond to the co-ordinates of the centre of the circular orbit up to a factor. The magnetic translation operators commute with \dot{x} and \dot{y} . However, they do not commute mutually.

$$[k_x, k_y] = -ieB. \quad (23)$$

So we can write down another independent set of creation and annihilation operators [5]

$$a = \sqrt{\frac{1}{2m\omega}} (-k_y - ik_x) \quad (24)$$

$$a^+ = \sqrt{\frac{1}{2m\omega}} (-k_y + ik_x), \quad (25)$$

with the commutation relation

$$[a, a^+] = -1. \quad (26)$$

We choose the notation that a^+ and a are raising and lowering operators for the angular momentum J ,

$$[J, a^+] = a^+, \quad (27)$$

$$[J, a] = -a. \quad (28)$$

In this convention a is the creation operator and a^+ is the annihilation operator for the magnetic quantum number of $k_x^2 + k_y^2$.

$$\frac{1}{2m} (k_x^2 + k_y^2) = \left(aa^+ + \frac{1}{2} \right) \omega. \quad (29)$$

Because the operators a and a^+ commute with the Hamiltonian they do not change the energy eigenvalue when they are applied to an energy eigenstate. It was found in [5] that the energy quantum number n for b^+b , the magnetic translation quantum number k for aa^+ and the angular momentum quantum number j for J are related by [5]

$$n = k + j \quad (30)$$

Hence, for the Laughlin wavefunctions which are also angular momentum eigenstates, the operator a lowers the angular momentum eigenvalue by one, but raises the magnetic translation eigenvalue by one leaving the energy unchanged.

The operators a and a^+ do not appear in the Hamiltonian. However, they are equally important as the operators b^+ and b . To illustrate this we shall present a calculation of the path integral of the Landau problem via these two sets of creation and annihilation operators. The method employed is just the computation of the coherent state wavefunctions. To discuss the calculation of wavefunction we need to choose a gauge for the magnetic field potential. We work in the symmetric gauge where

$$A_x = -\frac{1}{2}By; \quad A_y = \frac{1}{2}Bx. \quad (31)$$

In this section we shall set the cyclotron frequency ω as well the mass m of the electron to unity in order to express the formulae in a more concise form. In this gauge the relevant operators take the form

$$a = \sqrt{\frac{1}{2}} (-2\partial_{\bar{z}} + \frac{1}{2}z), \quad (32)$$

$$a^+ = \sqrt{\frac{1}{2}} (2\partial_z + \frac{1}{2}\bar{z}), \quad (33)$$

$$b^+ = \sqrt{\frac{1}{2}} (-2\partial_z + \frac{1}{2}\bar{z}), \quad (34)$$

$$b = \sqrt{\frac{1}{2}} (2\partial_{\bar{z}} + \frac{1}{2}z), \quad (35)$$

$$J = \bar{z}\partial_{\bar{z}} - z\partial_z. \quad (36)$$

We work with wavefunctions $\langle z|n, k\rangle$ [2, 5, 8] which are eigenstates with eigenvalues n and k which are non-negative integers. The general expression is quite complicated, being represented by hypergeometric functions [9]. However, the wavefunction of the ground state of the number operators is quite simple in form

$$\langle z|0, 0\rangle = \frac{e^{-\frac{z\bar{z}}{4}}}{\sqrt{2\pi}}. \quad (37)$$

Now we can define the coherent state [10] for our problem,

$$|u, \xi\rangle = e^{ub^+ - \bar{u}b} e^{\xi a - \bar{\xi} a^+} |0, 0\rangle. \quad (38)$$

By the algebraic relations the overlap of different coherent states is given by

$$\langle u, \xi|v, \zeta\rangle = e^{-\frac{1}{2}|u|^2 - \frac{1}{2}|v|^2 + v\bar{u}} e^{-\frac{1}{2}|\xi|^2 - \frac{1}{2}|\zeta|^2 + \zeta\bar{\xi}}. \quad (39)$$

Explicitly we have the exact results

$$\langle z|u, \xi\rangle = e^{-\frac{1}{2}u\bar{u} + \frac{\sqrt{2}}{2}u\bar{z}} e^{-\frac{1}{2}\xi\bar{\xi} + \frac{\sqrt{2}}{2}\xi(z - \sqrt{2}u)} \left(\frac{e^{-\frac{1}{4}z\bar{z}}}{\sqrt{2\pi}} \right). \quad (40)$$

Just like the case of the harmonic oscillator the path integral for the problem can be recovered quite easily from the coherent-state wavefunctions by the definition of the finite time propagator,

$$\langle z'|e^{-iHt}|z\rangle = e^{-\frac{it}{2}} \langle z'|ve^{-it}, \zeta\rangle \langle v, \zeta|z\rangle, \quad (41)$$

where we sum over all the intermediate states v and ζ . It is straightforward to do the gaussian integration to get the result [11]

$$\frac{1}{4\pi i \sin(t/2)} \exp \left[\frac{i(z' - z)(\bar{z}' - \bar{z})}{4} \cot(t/2) + \frac{1}{4}(z'\bar{z} - z\bar{z}') \right]. \quad (42)$$

So we get the expression for the path integral quite effortlessly. The calculation follows from familiar techniques of the coherent state wavefunctions. The Landau problem is a linear oscillation problem with two frequencies. It has two degrees of freedom. So the zero modes must be considered for the completeness of the problem. Hence we need a coherent state wavefunction for b as well as a^+ to account for the zero mode degeneracy. In the next section we shall show that this representation in terms of two sets of creation and annihilation operators is also useful for solving the Landau problem in an external symmetric quadratic potential.

IV. Solution with a symmetric quadratic potential

We propose to study the Landau problem with an external symmetric quadratic potential

$$V(x, y) = \frac{1}{2} \epsilon m \omega^2 (x^2 + y^2). \quad (43)$$

This is solved exactly by Silverman [12] using right and left circularly polarised quanta in the Coulomb gauge. We work out the solution here by a new method employing two sets of creation and annihilation operators in a gauge independent way. The procedure exhibits that the quantum theory, akin to the classical problem, is a linear oscillator problem. As in [5] we replace the co-ordinates x, y by \dot{x}, \dot{y}, k_x and k_y and rewrite the Hamiltonian in terms of the two sets of raising and lowering operators a, a^+ and b^+, b .

$$H = (\epsilon a a^+ + (1 + \epsilon) b^+ b + \epsilon(a^+ b + a b^+) + \frac{1}{2} + \epsilon) \omega. \quad (44)$$

To diagonalise the Hamiltonian we can take linear combinations of the creation and annihilation operators

$$A = u a + v b, \quad (45)$$

$$A^+ = u a^+ + v b^+, \quad (46)$$

$$B^+ = u b^+ + v a^+, \quad (47)$$

$$B = u b + v a. \quad (48)$$

where u, v are real and positive coefficients. The operators A^+, B^+ and A, B are raising and lowering operators for the angular momentum J , respectively like the previous case. This form ensures that we have two sets of mutually commuting creation and annihilation operators. Of course, we must demand

$$[A, A^+] = -1, \quad (49)$$

$$[B, B^+] = 1. \quad (50)$$

We then find that,

$$u = \frac{\sqrt{1 + 4\epsilon + 1}}{2\sqrt[4]{1 + 4\epsilon}}, \quad (51)$$

$$v = \frac{\sqrt{1+4\epsilon} - 1}{2\sqrt[4]{1+4\epsilon}}. \quad (52)$$

The Hamiltonian is given by

$$H = (AA^+ + \frac{1}{2})\omega_1 + (B^+B + \frac{1}{2})\omega_2, \quad (53)$$

where ω_1 and ω_2 are the two frequencies in the classical case.

Since J commutes with AA^+ and B^+B , we can take a representation $\Psi_{\alpha,\beta}$ such that

$$(AA^+)\Psi_{\alpha,\beta} = \alpha\Psi_{\alpha,\beta}, \quad (54)$$

$$(B^+B)\Psi_{\alpha,\beta} = \beta\Psi_{\alpha,\beta}, \quad (55)$$

$$J\Psi_{\alpha,\beta} = j\Psi_{\alpha,\beta}, \quad (56)$$

and there is the relation

$$\beta = \alpha + j. \quad (57)$$

The energy levels are given by

$$E = (\alpha + \frac{1}{2})\omega_1 + (\beta + \frac{1}{2})\omega_2, \quad (58)$$

where $\alpha \geq 0$ and $\beta \geq 0$.

To write down the wavefunctions we must adopt a gauge. Then we can specify the ground state as

$$A^+\Psi_{0,0} = 0, \quad (59)$$

$$B\Psi_{0,0} = 0. \quad (60)$$

and the other excited states are obtained by the raising operators

$$\Psi_{\beta,\alpha} = \frac{B^{+\beta}}{\sqrt{\beta!}} \frac{A^\alpha}{\sqrt{\alpha!}} \Psi_{0,0}. \quad (61)$$

The state $\Psi_{\beta,\alpha}$ is an eigenstate of J with eigenvalue $\beta - \alpha$. In the following we shall work out explicitly the wavefunctions in the symmetric gauge where rotational symmetry is explicit. In the symmetric gauge the vector potential can be represented as

$$A_x = -\frac{1}{2}By; \quad A_y = \frac{1}{2}Bx. \quad (62)$$

It is convenient to work in terms of the complex co-ordinates. We now set the cyclotron frequency ω as well the mass m of the electron to unity in order to express the formulae in a more concise

form. The creation and annihilation operators take the form

$$A = \sqrt{\frac{1}{2}} \left(-\frac{2}{\sqrt[4]{1+4\epsilon}} \partial_{\bar{z}} + \frac{\sqrt[4]{1+4\epsilon}}{2} z \right) \quad (63)$$

$$A^+ = \sqrt{\frac{1}{2}} \left(\frac{2}{\sqrt[4]{1+4\epsilon}} \partial_z + \frac{\sqrt[4]{1+4\epsilon}}{2} \bar{z} \right) \quad (64)$$

$$B^+ = \sqrt{\frac{1}{2}} \left(-\frac{2}{\sqrt[4]{1+4\epsilon}} \partial_z + \frac{\sqrt[4]{1+4\epsilon}}{2} \bar{z} \right) \quad (65)$$

$$B = \sqrt{\frac{1}{2}} \left(\frac{2}{\sqrt[4]{1+4\epsilon}} \partial_{\bar{z}} + \frac{\sqrt[4]{1+4\epsilon}}{2} z \right). \quad (66)$$

The angular momentum operator takes the simple form

$$J = \bar{z} \partial_{\bar{z}} - z \partial_z. \quad (67)$$

The normalised ground state wavefunction is

$$\Psi_{0,0} = \frac{\sqrt[4]{1+4\epsilon}}{\sqrt{2\pi}} e^{-\frac{\sqrt{1+4\epsilon}}{4} z \bar{z}}. \quad (68)$$

It is easy to see that

$$\Psi_{\alpha,0} = \frac{\sqrt[4]{1+4\epsilon}}{\sqrt{2\pi}} \frac{(\sqrt[4]{1+4\epsilon} z)^\alpha}{\sqrt{2^\alpha \alpha!}} e^{-\frac{\sqrt{1+4\epsilon}}{4} z \bar{z}}, \quad (69)$$

$$\Psi_{0,\beta} = \frac{\sqrt[4]{1+4\epsilon}}{\sqrt{2\pi}} \frac{(\sqrt[4]{1+4\epsilon} \bar{z})^\beta}{\sqrt{2^\beta \beta!}} e^{-\frac{\sqrt{1+4\epsilon}}{4} z \bar{z}}, \quad (70)$$

and the other states can be obtained by successively applying appropriate creation operators on the ground state.

Analogously we can define the coherent state wavefunction for this problem and work out the path integral by gaussian integration. We expect to find results as in Cheng [13].

V. Conclusion

We have shown that the two oscillation modes of the Landau problem can be quantised. We have one kind of quanta with energy $\hbar\omega$ corresponding to the cyclotron motion of the electron in a circle. The other kind of quanta is of zero energy and corresponds to the translational symmetry of the centre of the circular orbit. In this way the Landau levels as well as the degeneracy are well accounted for. We can also apply techniques of solving the oscillator problem to the Landau problem. The path integral can then be derived in a simple way through the calculation of gaussian integrals via the concept of coherent state wavefunctions. We also employ similar techniques to investigate the Landau problem in a symmetric quadratic external field. We should note that the operators defined are independent of the gauge choice. Boundary effects are not discussed and

these are important for realistic problem. The use of complex co-ordinates is due to the rotational symmetry of the problem. Treatment of non-symmetric potential may call for different co-ordinate systems. Of course it is also important to extend this operator approach to the many body problem which is relevant for the quantum hall effects.

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