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# Stochastic Mechanics in Complex Space

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We show that a backward stochastic system with an imaginary diffusion coefficient in complex space is equivalent to a real space quantum system if it obeys a Newtonian-like dynamical law and its final spatial distribution is constrained to be in real space. The ensemble averages of the position, momentum, angular momentum and energy of the stochastic processes are the quantum mechanical expectation values of the corresponding operators. The product of the mean-square deviations of the position and momentum of the stochastic process agrees with the corresponding uncertainty relation of quantum mechanics. The equation of motion is invariant under the combined transformations of changing the time $t \to -t$ and taking the complex conjugate of the equation. The stochastic motion reduces to the classical motion in the limit where the de Broglie wavelengh of the particle vanishes.

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## I. Introduction

This work aims at developing a stochastic mechanical theory of quantum mechanics. The stochastic approach to quantum mechanics in real space is first developed by Nelson [1, 2]. Starting from the equation of motion for a classical particle with diffusion, the Schrödinger equation is derived. Approaches using control theory are developed subsequently [3]. The uncertainty relation [4, 5, 6] and interference phenomena [7, 8] can be properly interpreted within the stochastic theory. However, it suffers a fundamental problem when there are nodal surfaces in the amplitude of the wave function. The particle can not move across the nodal surface. This means that the spaces separated by the nodal surfaces are mutually exclusive regions for the particle. Accordingly, the different parts of the wave function separated by the nodal surfaces should evolve independently. But this is in general not true.

Nevertheless, it may still be possible to develop a stochastic theory of quantum mechanics in complex space because the nodal surfaces of the wave function in real space may be circumvented. Development along this line has been proposed by using control theory [9], although the physical meaning of a particle's trajectory in complex space is not clear. Recently, based on the weak measurement theory of quantum mechanics [10, 11], it is shown that a backward stochastic process in complex space can be associated with a quantum system prepared in a state $|\psi\rangle$ at time $t$ (preselected state) and conditioned to be in a position eigenstate $|x_f\rangle$ at a later time $t_f > t$ (postselected state) [12]. The final condition for the backward stochastic process is $x_f$.

The conditional expectation value of the stochastic trajectories in complex space is the weak value (ensemble-averaged value) of the position operator of the quantum system. The quantum mechanical expectation values of the position, momentum, angular momentum, and energy are the ensemble averages of the corresponding quantities of the stochastic process. The position-momentum uncertainty relation is the product of the mean square deviations of the position and momentum of the stochastic process.

Here we would like to show that, starting from a Newtonian-like equation of motion for a backward stochastic process with an imaginary diffusion coefficient in complex space the orthodox quantum mechanics can be obtained. The contents of this article are organized as follows: In Sec. II the backward stochastic process in complex space is introduced. The momentum, angular momentum, and energy of this stochastic process are defined in analogous to the classical mechanics. In Sec. III a Newtonian-like equation of motion is introduced for the backward stochastic process. The analytic property of the drift velocity together with the imaginary diffusion coefficient $\nu = i\hbar/2m$, where $\hbar$ is the Planck's constant divided by $2\pi$ and $m$ is the mass of the particle, lead to the equivalence between the Newtonian-like equation of motion in complex space and the Schrödinger equation in real space. In Sec. IV the ensemble averages of the position, momentum, angular momentum and energy of the stochastic process are shown to be the quantum mechanical expectation values of the corresponding operators if the final probability distribution of the stochastic process is constrained to be in real space. The product of the mean square deviations of the position and momentum of the stochastic process is the corresponding uncertainty relation of quantum mechanics. These correspondences enable us to interpret the absolute square of the wave function as the effective probability distribution in real space for the backward stochastic process in complex space. In Sec. V the condition of the classical limit is identified to be that the amplitude of the wave function does not change appreciably in the space of one de Broglie wavelength. Also, it is shown that the equation of motion is invariant under the combined transformations of changing the time $t \to -t$ and taking the complex conjugate of the equation. Finally, a summary is given in Sec. VI.

## II. Markov process in complex space

Let us consider a backward Markov process in a $n$ dimensional complex space (For simplicity we shall consider one particle system only.)

$$d*z = v_-(z;t)dt + d*W; \qquad (1)$$

constrained by the final condition $z(t_f) = x_f$. Here $z = z_1; z_2; \cdots; z_n$, $z_k = x_k + iy_k$, $d*z = z(t) - z(t - dt)$, $v_-(z;t)$ is a $n$ component analytic function of $z$, and $d*W = W(t) - W(t - dt)$ is a Brownian-type displacement with an $imaginary$ diffusion coefficient $\nu = i\hbar/2m$, where $m$ is the mass of the particle. The condition that the drift velocity $v_-$ is analytic guarantees that (1) is Markov. The process can be written in terms of the real and imaginary components

$$d*x = Re(v_-)dt + \sqrt{\frac{1}{2}}d*w; \qquad (2)$$

$$d*y = Im(v_-)dt + \sqrt{\frac{1}{2}}d*w; \qquad (3)$$

where $\mathrm{Re}(\cdot)$ and $\mathrm{Im}(\cdot)$ denote the real and imaginary parts of $(\cdot)$, respectively, and $d^{*}w$ is a Brownian-type displacement with the real diffusion coefficient $\hbar/2m$. The probability distribution $\rho(z;t)=\rho(x;y;t)$ and the transition probability density $P(z;t\,|\,z';t')=P(x;y;t\,|\,x';y';t')$ are positive definite real-valued functions of complex variables satisfying the relation

$$\rho(z;t)=\int dx'\,dy'\,P(z;t\,|\,z';t')\rho(z';t'); \tag{4}$$

at $t\le t'$. The evolution of the transition probability density satisfies the backward Fokker-Planck equation

$$\partial_t P+\partial_x[\mathrm{Re}(v_-)P]+\partial_y[\mathrm{Im}(v_-)P]+\frac{|\nu|}{2}(\partial_x^2+\partial_y^2)P+|\nu|\,\partial_x\partial_y P=0; \tag{5}$$

at $t\le t_f$. The probability distribution $\rho(z;t)$ at $t\le t_f$ also satisfies the same equation

$$\partial_t\rho+\partial_x[\mathrm{Re}(v_-)\rho]+\partial_y[\mathrm{Im}(v_-)\rho]+\frac{|\nu|}{2}(\partial_x^2+\partial_y^2)\rho+|\nu|\partial_x\partial_y\rho=0; \tag{6}$$

The momentum, angular momentum, and energy of this Markov process can be defined based on classical mechanics [12]. For deterministic motion, the velocity of a particle at a point on it's trajectory is defined as

$$v(x;t)=\lim_{\Delta t\to 0}\frac{\Delta x}{\Delta t};$$

where $x(t)$ is the trajectory of the particle. For stochastic motion, the above definition can not be directly applied to a single stochastic trajectory due to the existence of the random term. Nevertheless, an ensemble averaged velocity at every point of the trajectory can be defined. Using (1) the ensemble averaged velocity entering the point $z$ at the time $t$ is

$$\lim_{\Delta t\to 0^+}E\left(\frac{\Delta z}{\Delta t}\right)=v_-(z;t); \tag{7}$$

where $\Delta z=z(t)-z(t-\Delta t)$. This suggest that the momentum and angular momentum at a point $z$ on the trajectory can be defined as

$$p(z;t)=mv_-(z;t); \tag{8}$$

$$l(z;t)=z\times p(z;t): \tag{9}$$

The kinetic energy at every point on the trajectory can be defined in an analogous way [1]. Noting that it involves the square of $\Delta z$ and $\Delta W^2\propto\Delta t$, the contribution from the random term has to be treated properly. An integration of (1) yields

$$z(t-\Delta t)=z(t)+\int_t^{t-\Delta t}dr\,v_-(z(r);r)+W(t-\Delta t)-W(t): \tag{10}$$

This means that

$$\begin{aligned}\Delta z &= z(t) - z(t-\Delta t)\\ &= -\int_t^{t-\Delta t} dr\, v_-(z(r);r) + W(t) - W(t-\Delta t).\end{aligned}\tag{11}$$

Substituting (10) into the integral of (11), expanding $v_-$ with respect to the point $z(t)$ to the order $\Delta t$, and making use of the Ito calculus [13] the $k$th component of (11) can be expressed as

$$\Delta z_k = v_{-k}(z;t)\Delta t - \left(\nabla\cdot v_{-k}\int_t^{t-\Delta t} dr\,[W(t) - W(r)]\right) + \Delta W_k \tag{12}$$

This leads to the result [1]

$$\lim_{\Delta t\to 0^+} E\left(\left(\frac{\Delta z}{\Delta t}\right)^2\right) = v_-^2(z;t) - \frac{i\hbar}{m}\nabla\cdot v_-(z;t) + \lim_{\Delta t\to 0^+} i\frac{3\hbar}{2m\Delta t}. \tag{13}$$

The singular term in (13) is a constant that is the same for all trajectories. It can be removed from the energy by a proper choice of the zero point for the energy. This suggest that the kinetic energy can be defined as

$$\frac{p^2}{2m} = \frac{1}{2}mv_-^2 - \frac{i\hbar}{2}\nabla\cdot v_-;$$

or equivalently

$$p^2(z;t) = m^2v_-^2(z;t) - i\hbar m\nabla\cdot v_-(z;t); \tag{14}$$

and the energy is

$$E(z;t) = \frac{1}{2}mv_-^2(z;t) - \frac{i\hbar}{2}\nabla\cdot v_-(z;t) + V(z;t). \tag{15}$$

Here $V$ is the potential energy and it has to be an analytic function of $z$ if the momentum and energy are to be well defined for the complex stochastic process.

## III. Stochastic mechanics in complex space

The drift velocity $v_-(z;t)$ is determined by the dynamics of the backward stochastic process. We shall show that the analytic property of the drift velocity together with the imaginary diffusion coefficient lead to the equivalence between a Newtonian-like equation of motion in complex space and the Schrödinger equation in real space. The equation of motion will involve the time derivative. But the time derivative along a single stochastic trajectory does not exist. Nevertheless, a mean backward time derivative of an analytic function $G(z;t)$ for the backward stochastic process can be defined as

$$\begin{aligned}(D_-G)(z;t) &= \lim_{\Delta t\to 0^+}(\Delta t)^{-1}E[G(\xi(t);t) - G(\xi(t-\Delta t);t-\Delta t)]_{\xi(t)=z}\\ &= \partial_t G + v_-\cdot\nabla G - \nu\nabla^2 G.\end{aligned}\tag{16}$$

Here $D_-$ denotes the mean backward time derivative and $E[\cdot]_{\xi(t)=z}$ is the conditional expectation value of the quantity in the parenthesis along the backward stochastic trajectory with the constraint $\xi(t) = z$. Apparently, $D_- z = v_-(z;t)$. In the limit $\nu \to 0$ the mean backward time derivative reduces to the ordinary time derivative. We now make the hypothesis that the equation of motion for a particle undergoing the backward stochastic motion (1) is

$$mD_- D_- z = mD_- v_- = F; \tag{17}$$

where $F$ is the force acting on the particle. In the limit $\nu \to 0$ (17) reduces to the Netwon's second law. Since $v_-$ is analytic it can be obtained by solving the equation of motion (17) in real space. We shall show that, in real space, (17) is in effect the Schrödinger equation. To proceed we first note that the diffusion coefficient $\nu$ is imaginary and the drift velocity in real space can be written as

$$v_-(x;t) = \Gamma(x;t) + 2\nu\Lambda(x;t); \tag{18}$$

where $\Gamma(x;t)$ and $2|\nu|\Lambda(x;t)$ are the real and imaginary parts of $v_-(x;t)$. We next consider the equation of motion (17) in real space for the two cases $\nabla \times F = 0$ and $\nabla \times F \neq 0$, respectively.

(i) $\nabla \times F = 0$

This implies $\nabla \times v_- = 0$ and (18) can be written as

$$\begin{aligned} v_-(x;t) &= \frac{\nabla S(x;t)}{m} - 2\nu\nabla R(x;t); \\ &= \frac{\nabla S(x;t)}{m} - 2\nu\frac{\nabla f(x;t)}{f(x;t)}; \end{aligned} \tag{19}$$

where $S$, $R$, and $f$ are real functions. Substituting (19) into (17) and making use of (16) the equation of motion is

$$\begin{aligned} &\nabla\left\{\left[\partial_t S + \frac{1}{2m}(\nabla S)^2 + 2m\nu^2\frac{\nabla^2 f}{f}\right] - 2m\nu\left(\frac{\partial_t f}{f} + \frac{\nabla f}{f}\cdot\frac{\nabla S}{m} + \frac{1}{2m}\nabla^2 S\right)\right\} \\ &= -\nabla V; \end{aligned} \tag{20}$$

where the force $F$ is replaced by the gradient of a potential $V$. With $\nu = i\hbar/2m$ and by properly choosing the zero point of the potential $V$, (20) is nothing but the Schrödinger equation

$$i\hbar\partial_t\psi(x;t) = -\frac{\hbar^2}{2m}\nabla\psi(x;t) + V\psi(x;t)$$

where

$$\psi(x;t) = f(x;t)e^{iS(x;t)/\hbar}: \tag{21}$$

(ii) $\nabla \times F \neq 0$

This implies $\nabla \times v_- \neq 0$ and it can be written as

$$v_-(x;t) = \frac{\nabla S(x;t)}{m} - \frac{e}{mc}A(x;t) - 2\nu\frac{\nabla f(x;t)}{f(x;t)}; \tag{22}$$

where $e$ is the charge of the particle, $c$ is the velocity of light and $\nabla \times (eA/mc) = -\nabla \times v_-$. By identifying $A$ as the vector potential corresponding to the magnetic field $B$ and assuming the force

$$F = -\nabla V + eE + \frac{e}{c} v_- \times B, \tag{23}$$

where $V$ is the non-electromagnetic potential and $E$ is the electric field, the equation of motion (17) in real space is the Schrödinger equation [14]

$$i\hbar \partial_t \psi = \frac{\hbar^2}{2m} \left( \frac{\hbar}{i} \nabla - \frac{e}{c} A \right)^2 \psi + (e\phi + V)\psi$$

where $\phi$ is the scalar potential of the electromagnetic field.

## IV. Stochastic interpretation of quantum mechanics

In Sec. III we have shown that the drift velocity of the backward stochastic process (1) in complex space is determined by the Schrödinger equation in real space if the dynamics of the backward stochastic process is Newtonian-like. But this is not sufficient to say that there is a correspondence between a real space quantum system and a complex space backward stochastic system. To interpret the quantum effect as the complex space stochastic effect seen in real space we need to show that the physical quantities of the stochastic process have well defined meanings in quantum mechanics. In the following we shall show that the conditonal expectation values and the ensemble averages of the position, momentum, angular momentum, and energy defined in Sec. II for the backward stochastic process are the weak values [10, 11] and the expectation values, respectively, of the corresponding quantum operators provided that the final condition for $\varrho(z, t)$ is chosen to be $\varrho(z_f, t_f) = |\psi(x_f, t_f)|^2 \delta(y_f)$. For simplicity we shall consider the case that the vector potential is zero.

First, we note that the conditional expectation values of the backward stochastic process satisfying the dynamics (17) in complex space can be expressed in terms of complex quantities in real space. To see this we consider the backward Fokker-Planck equation in real space

$$\frac{\partial}{\partial t} \bar{P}(x, t \,|\, x_f, t_f) + \nabla \cdot [v_-(x, t) \bar{P}(x, t \,|\, x_f, t_f)] + \nu \nabla^2 \bar{P}(x, t \,|\, x_f, t_f) = 0, \tag{24}$$

where $t \le t_f$, $v_-$ is the drift velocity of the backward stochastic process satisfying (17), and $\nu$ is the same diffusion coefficient used in (1). With $v_-$ given by the solution of the Schrödinger equation through (19) and (21), the solution of (24) is

$$\bar{P}(x, t \,|\, x_f, t_f) = \frac{K(x_f, t_f, x, t)\psi(x, t)}{\psi(x_f, t_f)}, \tag{25}$$

where $K(x_f, t_f, x, t)$ is the propagator of the Schrödinger equation. Note that the two relations

$$\lim_{t \to t_f} \bar{P}(x, t \,|\, x_f, t_f) = \delta(x - x_f),$$

$$\int dx \bar{P}(x, t \,|\, x_f, t_f) = 1,$$

are satisfied. Defining

$$\langle g(x;t)\rangle_{x(t_f)=x_f} \equiv \int dx\, g(x;t)\bar{P}(x;t\,|\,x_f;t_f); \tag{26}$$

the boundary condition of the Schrödinger equation assures that $\langle x^n\rangle_{x(t)=x_f}$ exists if the integer $n$ is greater than certain value. At $t \le t_f$ (24) leads to the equation

$$\frac{d}{dt}\langle x^n\rangle_{x(t_f)=x_f} = n\langle x^{n-1}v_-(x;t)\rangle_{x(t_f)=x_f} - \frac{i\hbar}{2m}n(n-1)\langle x^{n-2}\rangle_{x(t_f)=x_f}. \tag{27}$$

Now consider the conditional expectation value $E(z^n)_{z(t_f)=x_f}$ of the backward stochastic process. Using (1) and Ito calculus [13], $E(z^n(t))_{z(t_f)=x_f}$ satisfies the equation

$$\begin{aligned}\frac{d}{dt}E(z^n(t))_{z(t_f)=x_f} &= nE(z^{n-1}(t)v_-(z;t))_{z(t_f)=x_f}\\ &\quad - \frac{i\hbar}{2m}n(n-1)E(z^{n-2}(t))_{z(t_f)=x_f};\end{aligned} \tag{28}$$

at $t \le t_f$. Since $v_-(z;t)$ is an analytic function of $z$ we can expand $v_-(z;t)$ in the power series of $z$. Then $\langle x^n\rangle_{x(t_f)=x_f}$ and $E(z^n)_{z(t_f)=x_f}$ will satisfy exactly the same coupled differential equations constrained by the same final condition $x_f^n$. It follows that $E(z^n)_{z(t_f)=x_f} = \langle x^n\rangle_{x(t_f)=x_f}$. If $g(z;t)$ is an analytic function of $z$ this equality assures that $E(g(z;t))_{z(t_f)=x_f} = \langle g(x;t)\rangle_{x(t_f)=x_f}$.

Next, we note that $\langle g(x;t)\rangle_{x(t_f)=x_f}$ is the weak value of the operator $g(\hat{x};t)$ resulting from the weak measurement of position performed at $0 \le t \le t_f$ on an ensemble of physical systems prepared (preselected) in the state $|\psi\rangle$ at time $t=0$ and conditioned to be (postselected) in the position eigenstate $|x_f\rangle$ at a later time $t_f$ [10, 11]. The weak measurement theory is developed by Aharonov, Albert, and Vaidman [10, 11] by considering a time-dependent interaction $H(t)$ of von Neumann type [15] between the system and the apparatus. They used the uncertainty principle as applied to the momentum and position of the "pointer" of the apparatus to show that, by sacrificing the accuracy of the measurement, the system can be made to be disturbed as little, or weakly, as possible in a measurement. The uncertainty in each individual weak measurement will of course be large but an accurate and meaningful result, called weak value, is nevertheless obtained when an ensemble average is taken. Specifically, for an ensemble of physical systems preselected in the state $|\psi\rangle$ at time $t=0$ and postselected in the state $|B\rangle$ at a later time $t_f$, the weak value for an observable $A$ in weak measurement made at time t, at $0 \le t \le t_f$, is

$$\langle A\rangle_{\mathrm{weak}} \equiv \frac{\langle B|\exp(-i\int_t^{t_f}H(t)\,dt)\,A\,\exp(-i\int_0^{t}H(t)\,dt)|\psi\rangle}{\langle B|\exp(-i\int_0^{t_f}H(t)\,dt)\,|\psi\rangle}. \tag{29}$$

This weak value is a complex quantity, whose real and imaginary parts correspond to the mean shifts in the position and momentum of the pointer, respectively. For $A = g(\hat{x};t)$ and $|B\rangle = |x_f\rangle$, it is straightforward to show that

$$\begin{aligned}\langle g(\hat{x};t)\rangle_{\mathrm{weak}} &= \int dx\, g(x;t)\bar{P}(x;t\,|\,x_f;t_f)\\ &= \langle g(x;t)\rangle_{x(t_f)=x_f}\\ &= E(g(z;t))_{z(t_f)=x_f}.\end{aligned} \tag{30}$$

Thus the conditional expectation value of an analytic function $g(z;t)$ for the backward stochastic process satisfying the dynamics (17) and constrained to be at the position $x_f$ at time $t_f$ is the weak value of the operator $g(\hat{x};t)$ for an ensemble of quantum systems preselected in the state $|\psi\rangle$ and postselected in the state $|x_f\rangle$. It follows from (30), (19), and (21) that the conditional expectation values of the momentum, angular momentum, and energy of the backward stochastic process defined in Sec. II are the weak values of the corresponding operators in quantum mechanics. Namely,

$$\begin{aligned} E(p(z;t))_{z(t_f)=x_f} &= \int dx\, m v_-(x;t)\bar{P}(x;t\,|\,x_f;t_f) \\ &= \langle\hat{p}\rangle_{weak}, \end{aligned} \tag{31}$$

$$\begin{aligned} E(z\times p(z;t))_{z(t_f)=x_f} &= \int dx\,[x\times m v_-(x;t)]\bar{P}(x;t\,|\,x_f;t_f) \\ &= \langle\hat{x}\times\hat{p}\rangle_{weak}, \end{aligned} \tag{32}$$

$$\begin{aligned} E(p^2(z;t))_{z(t_f)=x_f} &= \int dx\,[m^2 v_-^2(x;t) - i\hbar m\nabla\cdot v_-(x;t)]\bar{P}(x;t\,|\,x_f;t_f) \\ &= \langle\hat{p}^2\rangle_{weak}, \end{aligned} \tag{33}$$

$$\begin{aligned} E(E(z;t))_{z(t_f)=x_f} &= \int dx\left[\frac{1}{2}m v_-^2(x;t) - \frac{i\hbar}{2}\nabla\cdot v_-(x;t) + V(x;t)\right]\bar{P}(x;t\,|\,x_f;t_f) \\ &= \langle\hat{H}\rangle_{weak}. \end{aligned} \tag{34}$$

The quantum mechanical expectation value of **A** is the weighted average of it's weak value

$$\langle A\rangle = \sum_B |\langle B|\psi\rangle|^2\langle A\rangle_{weak}. \tag{35}$$

It follows from (4), (30), and (35) that, at $t \le t_f$ the ensemble average of $g(z;t)$ of the backward stochatic process is the quantum mechanical expectation value of $g(\hat{x};t)$ provided that the final distribution of the backward stochastic process is chosen to be $\rho(z_f;t_f) = |\psi(x_f;t_f)|^2\,\delta(y_f)$. Namely,

$$\begin{aligned} E(g(z;t)) &= \int dx\,dy\,g(z;t)\rho(z;t) \\ &= \int dx\,dy\,dx_f\,g(z;t)P(z;t\,|\,x_f;t_f)\,|\psi(x_f;t_f)|^2 \\ &= \int dx_f\,E(g(z;t))_{z(t_f)=x_f}\,|\psi(x_f;t_f)|^2 \\ &= \int dx_f\,\langle g(\hat{x};t)\rangle_{weak}\,|\psi(x_f;t_f)|^2 \\ &= \langle g(\hat{x};t)\rangle. \end{aligned} \tag{36}$$

Consequently, at $t \le t_f$, the ensemble averages of the momentum, angular momentum, and energy of the backward stochastic process are the quantum mechanical expectation values of the corresponding operators

$$\begin{aligned} E(p(z;t)) = E(mv_-(z;t)) &= \langle mv_-(\hat{x};t)\rangle \\ &= \langle \hat{p}\rangle; \end{aligned} \tag{37}$$

$$\begin{aligned} E(l(z;t)) = E(z \times mv_-(z;t)) &= \langle \hat{x} \times mv_-(\hat{x};t)\rangle \\ &= \langle \hat{l}\rangle; \end{aligned} \tag{38}$$

$$\begin{aligned} E(E(z;t)) &= E\left(\frac{1}{2}mv_-^2(z;t) - \frac{i\hbar}{2}\nabla\cdot v_-(z;t) + V(z;t)\right) \\ &= \left\langle \frac{1}{2}mv_-^2(x;t) - \frac{i\hbar}{2}(\nabla\cdot v_-)(x;t) + V(x;t)\right\rangle \\ &= \langle \hat{H}\rangle. \end{aligned} \tag{39}$$

The product of the mean-square deviations of the position and momentum of the stochastic process is the quantum mechanical position-momentum uncertainty relation

$$\begin{aligned} [E(z^2) - (E(z))^2][E(p^2) - (E(p))^2] &= (\langle\hat{x}^2\rangle - \langle\hat{x}\rangle^2)(\langle\hat{p}^2\rangle - \langle\hat{p}\rangle^2) \\ &\ge \frac{\hbar^2}{4}. \end{aligned} \tag{40}$$

Finally, we note that, though $\rho(z_f;t_f) = |\psi(x_f;t_f)|^2\delta(y_f)$, an explicit relation between $\rho(z;t)$ and $|\psi(x;t)|^2$ can not be obtained. Nevertheless (36) tells us that

$$E(g(z;t)) = \int dx\, g(x;t)\, |\psi(x;t)|^2. \tag{41}$$

This suggests that $|\psi(x;t)|^2$ can be interpreted as an effective probability distribution in real space for the complex backward stochastic process.

## V. Classical limit and time reversal

In the previous sections we have shown that a backward stochastic process in complex space is equivalent to a quantum system in real space provided that the dynamics of the backward stochastic process is Newtonian-like and it's final distribution is related to the quantum distribution by $\rho(z_f;t_f) = |\psi(x_f;t_f)|^2\delta(y_f)$. To complete the whole story we have to show that the classical motion will be recovered under certain limit. Also, the question whether the equation of motion (17) for the backward process is invariant under the time reversal transformation needs to be studied. First, let us discuss the classical limit. It is obvious that, in the limit $\epsilon \to 0$, the stochastic motion reduces to a deterministic motion and the equation of motion (17) reduces to the Newton's second law. However, $\epsilon$ is not a dimensionless quantity and it is meaningless to say that $\epsilon \to 0$ without refering to the physical meaning of this limit. To recover the classical motion the drift velocity $v_-$ has to reduce to the classical velocity $v_c$ and the imaginary part of (22) has to be vanishingly small compared with $v_c$. Since $|\nu| = \hbar/2m$ this means that $(\hbar/m)(\nabla f/f) \ll v_c$ or, equivalently, $\lambda_d(\nabla f/f) \ll 1$, where $\lambda_d$ is the de Broglie wavelength. That is, the classical motion will be recovered if the amplitude of the wave function does not vary appreciably in the space of one de Broglie wavelength. This is the physical meaning of $\nu \to 0$.

Next, let us discuss the question of time reversal transformation. Under the transformation $t \to -t$, the backward process (1) transforms into a forward process

$$dz = v_+(z;t)dt + dW; \tag{42}$$

where $dz = z(t+dt) - z(t)$, $v_+(z;t) = -v_-(z;-t)$, and $dW = W(t+dt) - W(t)$ has the same imaginary diffusion coefficient as (1). A mean forward time derivative of an analytic function $G(z;t)$ can be defined for the forward stochastic process

$$\begin{aligned}(D_+G)(z;t) &= \lim_{\Delta t \to 0^+} (\Delta t)^{-1} E[G(\xi(t+\Delta t); t+\Delta t) - G(\xi(t);t)]_{\xi(t)=z} \\ &= \partial_t G + v_+ \cdot \nabla G + \nu \nabla^2 G.\end{aligned} \tag{43}$$

It is clear that $D_- \to -D_+$ under the transformation $t \to -t$ and we have

$$D_- v_- \xrightarrow{t \to -t} D_+ v_+. \tag{44}$$

Thus the equation of motion (17) is not invariant under $t \to -t$ because the last term of the right hand side of (43) has a different sign from that of (16). However, since $\nu$ is imaginary, the complex conjugate of (43) will have the same form as (16). This means that if we define the time reversal transformation as the transformation $t \to -t$ followed by a complex conjugate transformation then the equation of motion (17) is time reversal invariant. This property is reflected in the time reversal transformation of the Schrödinger equation.

## VI. Summary and discussion

We have shown that, by introducing a Newtonian-like dynamical law and choosing a final probability distribution $\rho(z_f;t_f) = |\psi(x_f;t_f)|^2\delta(y_f)$, the backward stochastic system (1) in complex space is equivalent to a real space quantum system. The key to establish the equivalence is that the drift velocity is analytic and the diffusion coefficient is imaginary. The equation of motion is time reversal invariant if we define the time reversal transformation to be the combined transformation of changing $t \to -t$ and taking the complex conjugate of the equation. The ensemble averages of position, momentum, angular momentum, and energy of this stochastic process are the quantum mechanical expectation values of the corresponding operators, respectively. The product of the mean-square deviations of the position and momentum of the stochastic process is the quantum mechanical position-momentum uncertainty relation. The stochastic motion in complex space reduces to the classical motion in the limit where the amplitude of the wave function does not change appreciably in the space of one de Broglie wavelength.

This approach does not have the shortcoming of the previous approaches in real space [1, 3, 16]. But unlike the previous approaches the physical meaning of $|\psi(x;t)|^2$ is not trivial. Except at $t_f$, $|\psi(x;t)|^2$ does not have any direct relation with the probability distribution of the complex stochastic process. However, the equality between the ensemble averages of the physical quantities of the stochastic process and the expectation values of the corresponding operators of quantum mechanics suggests that $|\psi(x;t)|^2$ can be interpreted as an effective probability distribution in real space for the complex stochastic process. One question remains to be answered: Why it is the backward instead of the forward stochastic process that corresponds to the quantum system? A possible answer may lie on the fact that the quantum mechanics is a theory for a close system. A precision measurement changes the state of the system in an uncontrollable way and only the information of the system prior to the measurement can be obtained from the outcome of the measurement.

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## References

[1] E. Nelson, Phys. Rev. **150**, 1079 (1966); *Quantum Fluctuations* (Princeton University Press, Princeton, 1985).

[2] F. Guerra, Phys. Rep. **77**, 263 (1981).

[3] F. Guerra and L. M. Morato, Phys. Rev. D **27**, 1774 (1983); L. M. Morato, Phys. Rev. D **31**, 1982 (1985).

[4] L. De La Peña-Auerbach, and A. M. Cetto, Phys. Lett. A **39**, 65 (1972).

[5] D. de Falco, S. De. Martino, and S. De Siena, Phys. Rev. Lett. **49**, 181 (1982).

[6] S. Golin, J. Math. Phys. **26**, 2781 (1985).

[7] M. S. Wang, Phys. Rev. A **38**, 5401 (1988).

[8] N. C. Petroni, Phys. Lett. A **141**, 371 (1989).

[9] H. H. Rosenbrock, Proc. R. Soc. Lond. A **450**, 417 (1995); *ibid* **453**, 983 (1997).

[10] Y. Aharonov, D. Z. Albert, and Lev Vaidman, Phys. Rev. Lett. **60**, 1351 (1988).

[11] Y. Aharonov and L. Vaidman, Phys. Rev. A **41**, 11 (1990).

[12] M. S. Wang, Phys. Rev. Lett. **79**, 3319 (1997); Phys. Rev. A **57**, 1565 (1998).

[13] C. W. Gardiner, *Handbook of Stochastic Methods* (Springer-Verlag, Berlin, 1985).

[14] In the derivation $\nabla \times B = 0$ is assumed. This is because the external current and the particle can not occupy the same place.

[15] J. von Neumann, in *Quantum Theory and Measurement*, eds. J. A. Wheeler and W. H. Zurek (Princeton University Press, Princeton, 1983).

[16] C. L. Wu, Chin. J. Phys. **35**, 866 (1997).