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Symmetric Teleparallel General Relativity

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General relativity can be presented in terms of other geometries besides Riemannian. In particular, teleparallel geometry (i.e., curvature vanishes) has some advantages, especially concerning energy-momentum localization and its "translational gauge theory" nature. The standard version is metric compatible, with torsion representing the gravitational "force". However there are many other possibilities. Here we focus on an interesting alternate extreme: curvature and torsion vanish but the nonmetricity ∇g does not-it carries the "gravitational force". This *symmetric teleparallel* representation of general relativity covariantizes (and hence legitimizes) the usual coordinate calculations. The associated energy-momentum density is essentially the Einstein pseudotensor, but in this novel geometric representation it is a true tensor.

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I. Introduction

Alternate representations of a theory often provide valuable insight. For Einstein's gravitational theory, general relativity (GR), there are alternate variables [1,2,3] and even alternate geometries [4]. The standard representation of GR has a Riemannian geometry with the curvature representing the "gravitational tidal force". In contrast, for the alternate geometries considered here, the curvature *vanishes*.

Such *teleparallel* (a.k.a. absolute parallel, fernparallelismus, Weitzenböck) geometries [5,6] have a venerable history. Einstein himself used such a geometry in one of his unified field theory attempts [7]. Others have used it to formulate alternate gravity theories, e.g., [8]. Of interest here, however, is that GR can be recast in terms of teleparallel geometry, see e.g., [9,10]. Of late, such a formulation has attracted renewed attention, including [2,11-15]. A major advantage concerns energy-momentum, its representation, positivity and localization [9,11,13]. Another appeal is that the teleparallel formulation can be regarded as a "translational gauge theory" [10,13-15].

For a teleparallel geometry there is a preferred class of frames, which greatly simplify computations. They can be obtained by selecting any frame at one point and parallel transporting it to all other points. Since the curvature vanishes, the parallel transport is path independent so the resultant frame field is globally well defined-if the manifold is parallelizable (a strong global topological restriction but highly desirable physically-being necessary for spinor fields and the Cauchy problem). In such a teleparallel frame field the connection coefficients vanish. Teleparallel frame fields are unique up to a global (constant, rigid) linear transformation. Parallel transport in a teleparallel frame field is simply accomplished by keeping the coefficients of tensors constant, and the covariant derivative simply reduces to the partial derivative.

The standard teleparallel representation of GR has a metric compatible connection. While the curvature vanishes the torsion does not; it acts like a gravitational "force". The simplest description is in terms' of orthonormal-teleparallel frames. This representation is easily constructed: 'given the metric for GR, simply choose any orthonormal frame and declare it to be parallel (i.e., introduce a new connection, the one which has vanishing coefficients in this particular frame). Thus each Riemannian geometry is represented by not just one but rather a whole gauge equivalence class of teleparallel geometries! (The relevant geometric point has been nicely stated recently: "Strictly speaking, there is no such thing as curvature or torsion of spacetime, but only curvature and torsion of connections" [14].)

Such a teleparallel geometry was used by Møller [9] to define a gravitational energy-momentum tensor (not a pseudotensor). Geometrically it is a tensor-but the catch is that the whole geometrization is rotationally gauge dependent. This seeming liability of the orthonormal-teleparallel representation of GR has more recently proved advantageous: it has been exploited, via a suitable rotational gauge condition [16], to obtain a Hamiltonian based tensorial (not spinorial) positive energy proof [11].

The orthonormal-teleparallel representation of Einstein's theory is indeed interesting and useful (it has even been referred to as the *teleparallel equivalent of general relativity* [13]). However there are other possibilities which, as far as we can ascertain, have been generally overlooked (aside from a few hints like [17]§5.9 and [18]p32).

One could consider a general-non-metric compatible, non-symmetric-teleparallel connection (this is a special case of the Metric Affine type of geometry, see [17] fj5.9 and [6] p142). Each possibility can be simply represented by a choice of frame and vanishing connection coefficients. Thus general relativity has a great many teleparallel representations. There may even be some nice choices which have the torsion carrying part of the gravitational force and the non-metricity the rest. However, here we focus on just one interesting extreme: the *symmetric teleparallel* formulation of general relativity (STGR) in which the torsion vanishes but the nonmetricity Vg does not (this is just the opposite of the usual choice).

Geometry is determined by the metric and the connection. The latter is conveniently represented by the one-form coefficients determined by the covariant differential $\nabla e_\beta = e_\alpha \Gamma^\alpha{}_\beta$, where e_α is a general frame field with dual coframe ϑ^α. They determine the *curvature* $R^\alpha{}_\beta := d\Gamma^\alpha{}_\beta + \Gamma^\alpha{}_\gamma \wedge \Gamma^\gamma{}_\beta$ and *torsion* $T^\alpha = \frac{1}{2} T^\alpha{}_{\mu\nu} \vartheta^\mu \wedge \vartheta^\nu := d\vartheta^\alpha + \Gamma^\alpha{}_\gamma \wedge \vartheta^\gamma$ two-forms. The connection can be decomposed into the Levi-Civita connection and deformation one-forms:

$$\Gamma^\alpha{}_\beta = \Gamma^{\{\}\alpha}{}_\beta - A^\alpha{}_\beta. \tag{1}$$

The general decomposition of the deformation ([6]p141,[17]§3.10),

$$A_{\alpha\beta} = K_{\alpha\beta} - \frac{1}{2} Q_{\alpha\beta} - Q_{\gamma[\alpha\beta]} \vartheta^\gamma, \tag{2}$$

includes the contortion, $K_{\alpha\beta} = K_{\alpha\beta\gamma}\vartheta^\gamma$, which is linear in the torsion,

$$K_{\alpha\beta\gamma} = \frac{1}{2}(T_{\beta\alpha\gamma} - T_{\alpha\beta\gamma} + T_{\gamma\alpha\beta}), \tag{3}$$

and nonmetricity $Q_{\alpha\beta} = Q_{\alpha\beta\gamma}\vartheta^\gamma = -Dg_{\alpha\beta}$. It induces an associated decomposition **of** the curvature:

$$R^\alpha{}_\beta \equiv R^{\{\}\alpha}{}_\beta - D^{\{\}}A^\alpha{}_\beta + A^\alpha{}_\gamma \wedge A^\gamma{}_\beta. \tag{4}$$

An advantage of the teleparallel formulation appears in the variational principle. The standard Einstein-Hilbert scalar curvature Lagrangian density is asymptotically 0($1/r^3$), so the action diverges. This can be improved to a convergent $O(1/r^4)$ by removing a total derivative (leaving the field equations unchanged) but the resulting density is not covariant — the Noether arguments then give energy-momentum pseudotensors. In contrast, for the teleparallel formulations, removal of the total derivative leaves a covariant, asymptotically convergent, Lagrangian which generates a covariant energy-momentum tensor.

It is convenient to introduce the dual basis for forms: $\eta^{\alpha\beta\cdots} := *(\vartheta^\alpha \wedge \vartheta^\beta \cdots)$, in terms of which the scalar curvature decomposes as

$$\begin{aligned} R\eta = R^\alpha{}_\beta \wedge \eta^\beta{}_\alpha &= R^{\{\}\alpha}{}_\beta \wedge \eta^\beta{}_\alpha - D^{\{\}}A^\alpha{}_\beta \wedge \eta^\beta{}_\alpha + A^\alpha{}_\gamma \wedge A^\gamma{}_\beta \wedge \eta^\beta{}_\alpha \\ &= R^{\{\}}\eta - d(A^\alpha{}_\beta \wedge \eta^\beta{}_\alpha) - A^\alpha{}_\beta \wedge D^{\{\}}\eta^\beta{}_\alpha + A^\alpha{}_\gamma \wedge A^\gamma{}_\beta \wedge \eta^\beta{}_\alpha. \end{aligned} \tag{5}$$

But $D^{\{\}}\eta_\alpha{}^\beta$ vanishes (because the torsion and nonmetricity vanish for the Levi-Civita connection); hence, with $\Gamma^\alpha{}_\beta$ teleparallel, removing an exact differential from the Einstein-Hilbert scalar curvature Lagrangian 4-form, gives the (note: it's covariant) general teleparallel Lagrangian 4-form:

$$\mathcal{L} = A^\alpha{}_\gamma \wedge A^\gamma{}_\beta \wedge \eta^\beta{}_\alpha, \tag{6}$$

to which one can adjoin the vanishing curvature condition via a Lagrange multiplier.

The special case $T \neq 0 = Q$ has been studied extensively, e.g., [9-11,13-15], Here we examine the opposite case: $T = 0 \neq Q$. Then $\mathcal{L}$ becomes quadratic in Q. More generally there are 5 independent quadratic Q terms [19,20], and hence a whole 5-parameter class of symmetric teleparallel theories that really merit investigation. We expect that most would have new types of "gravitational" effects. A general investigation is underway; only the one special case equivalent to GR is considered here.

While the symmetry condition can be introduced into the action via a Lagrange multiplier, the simpler representation is in terms of a teleparallel frame where Γ vanishes. The symmetry condition then reduces to $T^\alpha = d\vartheta^\alpha = 0$. Consequently (at least locally) $\vartheta^\alpha = dx^\alpha$ for some coordinate system and covariant derivatives reduce to coordinate partial derivatives. The non-metricity tensor simplifies to $Q_{\mu\nu\lambda} = -g_{\mu\nu,\lambda}$, and the deformation tensor acquires the Christoffel symbol form:

$$A^\alpha{}_{\beta\gamma} = \frac{1}{2}g^{\alpha\lambda}(Q_{\beta\lambda\gamma} + Q_{\gamma\lambda\beta} - Q_{\beta\gamma\lambda}) = -\frac{1}{2}g^{\alpha\lambda}(\partial_\gamma g_{\beta\lambda} + \partial_\beta g_{\gamma\lambda} - \partial_\lambda g_{\beta\gamma}) = -\left\{ {\alpha \atop \beta\gamma} \right\}, \tag{7}$$

as expected from Eq. (1). Consequently the associated Lagrangian density becomes

$$\mathcal{L} = \sqrt{-g}g^{\mu\nu}\left(\left\{ {\alpha \atop \gamma\mu} \right\}\left\{ {\gamma \atop \nu\alpha} \right\} - \left\{ {\alpha \atop \gamma\alpha} \right\}\left\{ {\gamma \atop \mu\nu} \right\}\right), \tag{8}$$

which has often been used in the past. Notwithstanding appearances, we want to emphasize that, in terms of the STGR geometry this Lagrangian-and indeed a host of apparently non-covariant calculations done in numerous works especially in the early days-are viewed as covariant. Consider just one example: Tolman's calculation of the energy-momentum density for gravity [21]. The result, the Einstein pseudotensor, from the STGR viewpoint is a covariant object. The coordinate dependence of the energy-momentum density is now elevated from a choice of reference frame to a "gauge" choice of geometry. Thus the associated energy-momentum density is a covariant (but geometry gauge dependent) tensor.

This novel representation, in a kind of tour de force, transforms all the usual coordinate calculations, rendering them into a geometrically covariant form and thereby bestowing upon them a greater degree of legitimacy. In our opinion, the result is not merely cosmetic. Transforming the usual coordinate invariance into a gauge invariance associated with a connection of essentially the same type as used in other gauge theories invites a new teleparallel approach to gravity as a gauge theory-one based on a symmetric connection and nonmetricity rather than on metric compatibility and torsion. The STGR formulation brings a new perspective to bear on GR. We don't yet know what might be revealed. Certainly the STGR representation offers a convenient way to formally treat gravity like other fields. The gravitational interaction effects, via the nonmetricity, have a character much like a Newtonian force and are derived from a potential, the metric; nevertheless the formulation is geometric and covariant.

Of course the STGR formulation has some liabilities. It must be emphasized that in this geometry it is no longer possible to simply commute derivatives and the raising or lowering of indices via the metric as we are so accustomed to do in the standard Riemannian approach. Hence tensorial equations will appear differently depending on how the indices are arranged. However this apparent computational complication is to a large extent offset by the fact that in the teleparallel geometry covariant derivatives commute (since curvature vanishes). It should also be noted that the coupling to sources which effectively reproduces the standard GR interaction is *not* teleparallel minimal coupling. (This serves to draw attention to the fact that minimal coupling is not a sacred principle.) Another obvious limitation of the STGR formulation is that it (almost) requires a global coordinate system,

Of course many investigators have treated gravity as a field theory in flat spacetime without developing a teleparallel geometric formulation (see, e.g., Ch VI. §5 in [22] and Ch. 11 in Thorne's popular book [23]; the latter includes an interesting discussion of the philosophical issues). The STGR alternative offers our old familiar theory a new symmetric teleparallel apparel in which it acquires a different geometric appearance.

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