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E(2) Invariant Gauge Fields on the Plane

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We use the Lie derivative method to find solutions of E(2) invariant gauge fields on the plane. The gauge groups U(1) and $SU(2)$ are considered. Nontrivial field configurations are obtained.

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I. Introduction

Symmetric field configurations are always appealing. The Coulomb electric field of a point charge is probably the first example we encountered. The magnetic monopole field is also spherically symmetric. However, the vector potential of a magnetic monopole [1] does not look spherically symmetric at a first glance. It is in fact spherically symmetric modulo a gauge transformation. There is a subtle interplay between spacetime symmetries and internal gauge symmetries. Systematic studies of this subject were initiated by Forgacs and Manton [2]. There is a pedagogical review by Jackiw [3]. It was found that if there are several external symmetries not commuting, the consistency conditions impose severe restriction on the solutions of the symmetry equations. This implies that there are only a small number of solutions of the symmetry equations. The most extensively studied symmetry group is the rotational group SO(3). There is a closely related symmetry group, the E(2) group or the Euclidean group on the plane [4]. This is a semidirect product of the translation group on the plane and the rotation group about the z-axis. Indeed, the E(2) group is related to the rotation group SO(3) by a limiting process known as the contraction method. Consider a point $\boldsymbol{P}$ close to the north pole of a sphere of very large radius and the tangent plane to the sphere at the north pole. Rotation operations are reduced to E(2) group motions on the tangent plane. In this paper we propose to study E(2) spacetime symmetries of gauge fields on the plane, following the analysis of Fung [5].

This paper is organised as follows. In section 2 we give a brief review of the mathematical apparatuses needed – the Lie derivative, the symmetry equations for the vector potential and the consistency equations of the gauge functions. A cursory discussion of the Euclidean group E(2) will be given in section 3. In section 4 we attempt to find E(2) invariant solutions for the Maxwell U(1) field. The result is just the well-known constant magnetic field on the plane. In section 5 we give some solutions of E(2) invariant solutions of the $SU(2)$ gauge fields. In section 6 we make the final conclusion as well as elaborate on more discussions.

II. Lie derivatives, symmetry equations and the consistency equations

We consider an infinitesimal transformation

$$x'^{\mu} = x^{\mu} + \epsilon X^{\mu}, \tag{1}$$

generated by the vector field X^{μ} represented by the linear operator

$$X = X^{\mu}\frac{a}{\partial x^{\mu}}. \tag{2}$$

We have geometrical and physical objects represented by tensor fields. Under the infinitesimal transformation generated by X it is possible to define the Lie derivative of tensor fields in the direction of the field X. The Lie derivative transforms tensors to tensors of the same type. The formulae for the Lie derivatives of tensors are quite complicated. However, the formula for the Lie derivative L_X is especially simple when acting on differential forms, which is known as the homotopy formula [6]

$$i_X d + d i_X = L_X, \tag{3}$$

where i_X is the contraction operator and d is the exterior differential operator.

Next we consider a gauge field configuration represented by a matrix one-form

$$A = A_{\mu} dx^{\mu}, \tag{4}$$

where the gauge potential A,, is expressed in terms of the generators T^a of the gauge group G as

$$A,, = A^a_{\mu} T^a. \tag{5}$$

The generators T^a satisfy commutation relations designated as

$$[T^a, T^b] = f^{ab}_c T^c. \tag{6}$$

The field strength F is a two-form

$$F = dA + A^2. \tag{7}$$

Under a gauge transformation the gauge potential A transforms imhomogeneously as

$$A' = A + g^{-1} Dg, \tag{8}$$

where g is an element of the gauge group G and D is the covariant derivative on the matrix,

$$Dg = dg + [A, g]. \tag{9}$$

The field strength F, however, transforms covariantly as

$$F' = g^{-1} F g. \tag{10}$$

For infinitesimal gauge transformation, $g = 1 + \epsilon W$, the change of A is

$$A' = A + \epsilon DW. \tag{11}$$

Because of the gauge degree of freedom the gauge potential A is invariant under a spacetime transformation induced by a vector field X^μ if its Lie derivative with respect to X is zero up to a gauge transformation. The symmetry equations can be written as

$$L_X A = DW_X. \tag{12}$$

This interplay of spacetime symmetry and internal gauge symmetry was observed early by Bergmann and Flaherty [7]. However, they only considered one single symmetry. A new arena opens up if one considers [2,3] several non-commuting symmetries. Consistency conditions impose constraints on the gauge functions W_X. The consistency equations take the form

$$L_X W_Y - L_Y W_X - [W_X, W_Y] = W_{[X,Y]}. \tag{13}$$

Surprisingly, these consistency equations impose severe constraints on the possible solutions of A. The solutions of possible symmetrical gauge fields are usually quite interesting and physically relevant.

III. Euclidean group on the plane

The best known spacetime symmetry group is the rotation group **SO(3) in three** dimensions. Rotational transformation is effected by the three generators

$$X = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \tag{14}$$

$$Y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \tag{15}$$

$$Z = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}. \tag{16}$$

Discussions of SO(3) symmetric gauge fields are presented fully in [2,3,5].

The Euclidean group on the plane E(2) consists of three generators - translations along x and y directions plus rotation about z-axis. Indeed, the Euclidean group on the plane is obtained as a limit on the rotation in three dimensions. Consider a point P on the tangent plane of the north pole of a sphere of very large radius. Rotations in three dimensions will give Euclidean motions on the tangent plane. The three generators of the Euclidean group on the plane are thus given by

$$P_y = -\frac{\partial}{\partial y}, \tag{17}$$

$$P_x = \frac{\partial}{\partial x}, \tag{18}$$

$$M = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \tag{19}$$

with commutation relations given by

$$[P_y, P_x] = 0, \tag{20}$$

$$[P_x, M] = -P_y, \tag{21}$$

$$[M, P_y] = -P_x. \tag{22}$$

We have retained the order of the generators so that the relations between the E(2) group and the SO(3) are transparent. We also give here the three generators P_y, P_x and M in polar coordinates ρ, ϕ as follows,

$$P_y = -\sin\phi\frac{\partial}{\partial\rho} - \frac{\cos\phi}{\rho}\frac{\partial}{\partial\phi}, \tag{23}$$

$$P_x = \cos\phi\frac{\partial}{\partial\rho} - \frac{\sin\phi}{\rho}\frac{\partial}{\partial\phi}, \tag{24}$$

$$M = \frac{\partial}{\partial\phi}. \tag{25}$$

The translations and the rotation form two subgroups of the E(2) group. But the two subgroups do not commute with each other. They form a semidirect product. Hence the discussions on the E(2) are non-trivial and somewhat, a little bit different from the $SO(3)$ group.

IV. The Maxwell U(1) field

In this section we endeavour to find E(2) invariant configurations for the U(1) electromagnetic field. The solution is just the expected constant magnetic field perpendicular to the plane. Our calculations follow those of [5] which employed formulations in terms of differential forms. A good reference for the use of differential forms in physics is the book by Flanders [8].

The symmetry equations read as

$$L_X A = dW_X, \tag{26}$$

where A is a scalar one-form and W_X is a scalar function and the consistency equations reduce in the abelian case to

$$L_X W_Y - L_Y W_X = W_{[X,Y]}, \tag{27}$$

for non-commuting vector fields.

We shall work in polar coordinates. By a gauge transformation we can always take one of the W_X to vanish. We thus take $W_M = 0$. By means of the symmetry equation

$$L_M A = 0, \tag{28}$$

we can write

$$A = A_\rho(\rho)d\rho + A_\phi(\rho)d\phi. \tag{29}$$

The A_ρ term can be gauged away and we are left with the relevant A_ϕ term.

The consistency equations demand

$$L_M W_{P_y} = -W_{P_x}, \tag{30}$$

$$L_M W_{P_x} = W_{P_y}. \tag{31}$$

The solutions are easily obtained to be

$$W_{P_y} = f(\rho)\cos\phi, \tag{32}$$

$$W_{P_x} = f(\rho)\sin\phi, \tag{33}$$

where f(p) is a function waiting to be determined.

Now we can substitute the above expression in

$$L_{P_y} A = dW_{P_y}, \tag{34}$$

to get

$$-\frac{dA_\phi}{d\rho} + \frac{A_\phi}{\rho} = -f, \tag{35}$$

$$\frac{A_\phi}{\rho^2} = \frac{df}{d\rho}. \tag{36}$$

This admits two solutions. The function $f(\rho)$ can be C/ρ where C, is a constant. Then $A_\phi = -C$ which is trivial since the field strength vanishes. Another solution is to take $f(\rho)$ to be $C\rho$ and the resulting A_ϕ equals $C\rho^2$. To conform to the usual notations we take C = B/2 where B is the usual magnetic field strength. The nontrivial E(2) invariant vector potential is given as

$$A = \frac{B}{2}\rho^2 d\phi, \tag{37}$$

$$= \frac{B}{2}(xdy - ydx), \tag{28}$$

which is just the famous constant magnetic field potential in the symmetrical gauge. With explicit invariance for P_x, that is putting $W_{P_x} = 0$ *we* can get the potential in the Landau gauge.

V. Non-abelian SU(2) gauge fields

We now seek nontrivial E(2) invariant $SU(2)$ gauge field configurations. As before we impose explicit ϕ invariance, i.e. we take W_M to be zero. The gauge field potential A takes the form

$$A = A_t(t)dt \ \text{t} \ A_\rho(\rho,t)d\rho \ \text{t} \ A_\phi(\rho,t)d\phi, \tag{39}$$

where A_t, A,, and A_ϕ are matrices. Again we can evaluate the remaining W_{P_y} and W_{P_x},

$$W_{P_y} = f(\rho)\Omega\cos\phi, \tag{40}$$

$$W_{P_x} = f(\rho)\Omega\sin\phi, \tag{41}$$

where Ω is a matrix independent of ρ and ϕ and $f(\rho)$ is a scalar function to be determined. The gauge potential component $At(t)$ must commute with Ω.

The symmetry equation

$$L_{P_y}A = dW_{P_y} + [A, W_{P_y}], \tag{42}$$

gives rise to the following two equations,

$$-\sin\phi\frac{\partial A_\rho}{\partial\rho} + \frac{\cos\phi}{\rho^2}A_\phi = \frac{df}{d\rho}\cos\phi\Omega \ \text{t} \ f\cos\phi[A_\rho,\Omega], \tag{43}$$

$$-\sin\phi\frac{\partial A_\phi}{\partial\rho} - \cos\phi A_\rho + \frac{\sin\phi}{\rho}A_\phi = -f\sin\phi\Omega + f\cos\phi[A_\phi,\Omega]. \tag{44}$$

With hints from the abelian case we can make two choices. We can take $f(\rho)$ to be l/p and put

$$A_\phi = -\Omega + A;. \tag{45}$$

The above two equations simplify to

$$-\sin\phi\frac{\partial A_\rho}{\partial\rho} + \frac{\cos\phi}{\rho^2}A'_\phi = \frac{\cos\phi}{\rho}[A_\rho,\Omega], \tag{46}$$

$$-\sin\phi\frac{\partial A'_\phi}{\partial\rho} - \cos\phi A_\rho + \frac{\sin\phi}{p}A'_\phi = \frac{\cos\phi}{\rho}[A_\phi, R]. \tag{47}$$

The equations are consistently satisfied if we can make them independent of ρ and ϕ. Hence we can choose

$$A'_\phi = \rho\Omega_\phi, \qquad A_\rho = \Omega_\rho, \tag{48}$$

where Ω_ρ and Ω_ϕ are matrices satisfying

$$[\Omega_\phi,\Omega] = -\Omega_\rho, \tag{49}$$

$$[\Omega_\rho,\Omega] = \Omega_\phi. \tag{50}$$

In particular we can choose $\Omega = i\sigma_3$, and then we have

$$\Omega_\rho = \Phi_1 i\sigma_1 - \Phi_2 i\sigma_2, \tag{51}$$

$$\Omega_\phi = \Phi_2 i\sigma_1 + \Phi_1 i\sigma_2, \tag{52}$$

where Φ_1 and Φ_2 are arbitrary functions of t.

For the other case we can take f(p) to be ρ and set

$$A_\phi = \rho^2\Omega + A'_\phi. \tag{53}$$

However, it is easy to show that the symmetry equations admit only trivial solution,

$$A, = 0, \tag{54}$$

$$A'_\phi = 0, \tag{55}$$

So in this case we only have abelian solution.

We would like to add a remark. When A_t is identically equal to zero, we have the purely magnetic configuration. The first solution from equations (45,48) is an abelian solution even though it looks like non-abelian. This can be easily checked by working out the expression for its field strength. This means that there are no purely magnetic non-abelian configurations which is E(2) invariant. When A_t is nonzero, we get truly non-abelian solution which is a mixed magnetic and electric field configuration.

VI. Conclusion

We have obtained E(2) invariant gauge field solutions for the U(1) and $SU(2)$ cases using methods similar to those of obtaining SO(3) invariant solutions in [3, 5]. On the whole the results are similar to those of the SO(3) cases. However, the abelian case is a nontrivial limit of the spherically symmetric cases, since taking direct limit will yield zero field solution. The constant magnetic field solution is the term in the second order expansion. It is surprising that this solution is compatible with the consistency and symmetry equations. For the non-abelian case, purely magnetic non-abelian configuration does not occur because we only have magnetic field perpendicular to the plane. Non-abelian configurations appear for mixed electric and magnetic fields. These configurations may be relevant for studies in condensed matter physics just as the constant magnetic field yields the Landau problem which is ubiquitous in the studies of quantum hall effects.

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