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Vortex Dynamics, Resistivity Formula, and Fluctuation-dissipation Theorems

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We investigate the problem of forces on moving vortex in a superfluid or superconductor. The main purpose is to locate the source which leads to the contradictory results in the literature. We establish the connection between this problem to the difficult but well studied subject of resistivity formula in transport theory. The relaxation time approximation used in the force calculation via force-force correlation function is shown to be invalid. The roles of Berry's phase and fluctuation-dissipation theorems are discussed.

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I. Introduction

For a vortex moving in a superfluid or superconductor, the only agreement we have reached so far is at zero temperature, in absence of any impurity potentials. Question arises in more complicated cases. Using the Berry's phase calculation [1], as well as the exact total force-force correlation calculation [2], It has been shown that the transverse force is the Magnus force, coming from extended states with no dependence on the core level spacing and other extrinsic details. Other microscopic derivations using core state transitions [3-7] have shown, however, that the transverse force on a moving vortex is greatly reduced in magnitude when the core level spacing is less than the energy scale associated with random scattering. Apparently, it seems that they are not two equivalent ways to obtain the sarne quantity but rather at least one of them must be incorrect or incomplete. Thus if we could settle the difference on the transverse force on moving vortex between core or extended states originated, we should be able to reach an agreement. However, these two ways of calculating the transverse force on moving vortex are shown to be equivalent to each other [8]. It is impossible for them to give different results unless there is a hidden mistake elsewhere.

The algebra in the derivation of vortex dynamics is either involved or abstract.. The repeating of those calculations is not the most efficient route to detect and understand the source of disagreements. Instead, we note that the vortices satisfy a classical Langevin equation with parameters determined microscopically. This equation has the same form as a classical electron moving in a magnetic field. Therefore general properties of Langevin dynamics, for instance the fluctuation-dissipation theorems, should be obeyed by vortices. In addition, our knowledge on electronic transport can be readily used to help understanding vortex dynamics. In particular, we will examine the relaxation time approximation and show it to be invalid in resistivity calculation via force-force correlations.

Let us consider a classical charged particle in a magnetic field obeying a generalized Langevin equation:

$$m\dot{u}_i(t) = -\int_{t_0}^{t} dt' \, \eta_{ik}(t-t')u_k(t') + K_i(t) + B\varepsilon_{ik}\, u_k(t) + f_i(t). \tag{1}$$

Here the index $i = x$ or y, the velocity of the particle $\mathbf{u}(t) = (u_x(t),\ u_y(t))$, m the mass, $\mathbf{K}(t) = (K_x(t),\ K_y(t))$ an external force, $\mathbf{f}(t) = (f_x(t),\ f_y(t))$ a random force which simulates the effect of the thermal reservoir. The Einstein convention of the repeated indices as summation has been used. $B\varepsilon_{ik}\ u_k(t)$ represents the transverse force, the Lorentz force $\mathbf{u}(t) \times \mathbf{B}$ in the Langevin equation with the magnetic field taken along the z-direction. The matrix $\eta(t-t') = \{\eta_{ij}\}$ represents frictions in both longitudinal and transverse directions of the particle motion. Its possible finite off-diagonal elements will change the effect of the original Lorentz force on the particle. In addition, we have

$$\begin{aligned} < f_i(t) > &= 0, \\ < u_i(t_o)f_j(t_o+t) > &= 0, t > 0, \\ < u_i(t_o)u_j(t_o) > &= \frac{k_B T}{m}\delta_{ij}. \end{aligned} \tag{2}$$

The second one is due to the causality and the last one is the equipartition theorem.

The problem of particle response to a perturbation can be formulated into two different ways. We can calculate the velocity of the particle while the applied force is given. Here the Hamiltonian of the particle is known. In such a case, it is equivalent to obtaining a conductivity formula. Otherwise we can consider a given trajectory of particle motion and calculate what is the applied force needed to maintain it. It is equivalent to obtaining a resistivity formula, i.e. calculating electric field needed to maintain the given current. The derivation of vortex dynamics belongs to the second kind, where we assume a steady motion of vortex and calculate what is the external force acted on the vortex. In electron transport, we have a choice to formulate the problem in either way. In vortex dynamics, we can only formulate in terms of resistivity because the effective vortex Hamiltonian is unknown and is exactly what we want to obtain.

The conductivity may be obtained by the Nakano-Kubo's formula, a calculation of velocity-velocity correlation function. It may also be obtained by solving Boltzmann equation in presence of an electric field [11]. Both of them are standard methods. The relaxation time approximation is a valid one in these cases.

The resistivity formula is much more confusing. Since 1960s, various resistivity formulae have been proposed and examined repeatedly [9-12]. In the derivation of vortex dynamics, total force-force correlation formalism has been explicitly shown by Šimánek to be the one used in various Green's function calculations of forces on moving vortex [4]. In addition, the force-balance theory has been used to calculate forces acting on a vortex [5].

Recently, Boltzmann equation has also been used [7]. In the literature of electron transport, all of the above methods have been used in obtaining a resistivity formula, although normally discussed without magnetic field. It has been demonstrated that the longitudinal resistivity calculated directly using total force-force correlation function is always zero in the zero frequency limit [10]. Thus any formula that allows to obtain finite friction, or longitudinal resistivity, directly from total force-force correlation function must be incorrect in this limit. The longitudinal resistivity formula using Boltzmann equation gives either 0/0 results or a formula trivially equivalent to reciprocal conductivity formula [11].

We will first briefly review conductivity formula. As expected, we will show that in such a case $< \mathbf{u}(t) >$ is determined by velocity-velocity correlation functions in the absence of $\mathbf{K}$. We will verify that indeed the relaxation time approximation is valid in such a calculation. Parallelly, we will derive a resistivity formula by assuming a given velocity $\mathbf{u}(t)$ and calculating the average force $< \mathbf{K} >$ which should be applied to sustain such a motion. We find that the problems involved in vortex dynamics become clear after careful study of this simple model.

Now let us put $\mathbf{K} = 0$ in Eq. (1) and calculate the velocity-velocity correlation function. Multiply Eq. (1) by $u_j(t_0)$ and take the ensemble average:

$$m < \dot{u}_i(t_0+t)u_j(t_0) >= - \int_0^t dt' \eta_{ik}(t-t') < u_k(t_0+t')u_j(t_0) >$$
$$+ B\varepsilon_{ik} < u_k(t_0+t)u_j(t_0) > + < f_i(t_0+t)u_j(t_0) > .$$

The term $< u_i(t_0)f_j(t_0+t) >$ vanishes according to Eq. (2).

Introducing a Laplace transform

$$\eta[\omega] = \int_0^\infty dt\, e^{-i\omega t}\eta(t),$$

defining the velocity-velocity correlation function matrix

$$\mathcal{U}_{ij}(t) =< u_i(t_0+t)u_j(t_0) >, \tag{3}$$

and integrating by part, we have

$$\mathcal{U}[\omega] = (i\omega + \eta[\omega] D i\sigma_y B)^{-1} k_B T. \tag{4}$$

We have used the identity $m\mathcal{U}(0) = k_B T$. $\sigma_y = -i\{\varepsilon_{ij}\}$ is the Pauli matrix.

Next we calculate the total force-force correlation function matrix

$$\mathcal{F}_{ij}(t) = m^2 < \dot{u}_i(t_0+t)\dot{u}_j(t_0) > .$$

Taking the Laplace transform, using the translational invariant in time

$$< u_i(t_0+t)\dot{u}_j(t_0) >= - < \dot{u}_i(t_0+t)u_j(t_0) >,$$

and the total force-velocity correlation function

$$m < \dot{u}_i(t_0+t)u_j(t_0) > [\omega] = -m\mathcal{U}_{ij}(0) + im\omega\mathcal{U}_{ij}[\omega],$$

we have

$$\mathcal{F}[\omega] = (iB\sigma_y + \ imw \ + (m\omega)^2(im\omega \ t \ \eta[\omega] - i\sigma_y B)^{-1})k_B T. \tag{5}$$

In the limit w $\rightarrow$ 0, we have

$$\mathcal{F}[0] = iB\sigma_y k_B T, \tag{6}$$

which is independent of η.

We then calculate the random force-force correlation matrix

$$\mathcal{R}_{ij}(t) = < f_i(t_0 \ \text{t} \ t) f_j(t_0) > . \tag{7}$$

From Eq. (1) we can express R(t) in terms of total force-force, total force-velocity, and velocity-velocity correlation functions. Taking the Laplace transform and integrating by part, we obtain

$$\mathcal{R}[\omega] = \eta[\omega] m\mathcal{U}(0) = \eta[\omega] k_B T, \tag{8}$$

or $\eta(t) = \mathcal{R}(t)/(k_B T)$. This is the 'second' fluctuation-dissipation theorem described by Kubo [9]. The generalized friction $\eta(t)$ is given by the random force-force correlation. Because the random force is determined by the thermal bath, $\eta(t)$ has no off-diagonal part if the random force-force correlation matrix has not. An important conclusion we can draw from here is that the even though there is no time-reversal symmetry in the particle motion, the frictional force is always longitudinal as long as the heat bath does not generate an off-diagonal element in the random force-force correlation function matrix. For example, this is the case for a charged particle dynamics described by a single relaxation time in Boltzmann equation.

Now we look for the connection between the correlation functions and the transport coefficients. First we consider the mobility. With an applied external force K(t) $= \overline{\mathbf{K}}[\omega]e^{i\omega t}$ the mobility $\mu[\omega]$ is defined by $< \mathbf{u}_i[\omega] >= \mathbf{u}_{ij}[\omega]\mathbf{K}_j[\omega]$. From Eq. (1) we immediately obtain the mobility

$$\mu[\omega] = (\text{imw} \ + \eta[\omega] - i\sigma_y B)^{-1}$$

in the limit $t_0 \rightarrow -\infty$. Using Eq. (4), the mobility is related to the velocity-velocity correlation function

$$\mu[\omega] = \frac{\mathcal{U}[\omega]}{k_B T}. \tag{9}$$

This is the 'first' fluctuation-dissipation theorem described by Kubo [9], equivalent to the Nakano-Kubo's formula for the electrical conductivity.

Next. we consider that the particle is moving at a given velocity $\overline{\mathbf{u}}(t)$ and find out what is the external force needed to sustain such a motion. It is equivalent to the calculation of resistivity if the particle is charged. From Eq. (1), we have the average force

$$< K_i[\omega] >= (imw \ + \ \eta[\omega] - i\sigma_y B)_{ij}\overline{\mathbf{u}}_j[\omega] \tag{10}$$

which is trivially equivalent to the reciprocal of conductivity formula. Obviously this process does not provide us an independent way of calculating resistivity.

However, if we are only interested in the average force $<\mathbf{K}>$ in a steady state motion: we do have an alternative resistivity formula. After taking $w \rightarrow 0$ and using Eqs.(6) and (8), Eq. (10) gives

$$< K_i > [0] = \frac{1}{k_B T}(\mathcal{R}_{ij}[0] - \mathcal{F}_{ij}[0])\overline{u}_j[0]. \tag{11}$$

Taking η to be a scalar (proportional to a unit matrix), the external force can take a more suggestive form,

$$< \overline{\mathbf{K}}[0] >= \eta[0]\overline{\mathbf{u}}[0] - \overline{\mathbf{u}}[0] \times \mathbf{B}, \tag{12}$$

where the longitudinal component depends on $\mathcal{R}[0]$, the random force-force correlation function, and the transverse component only on $\mathcal{F}[0]$, the total force-force correlation function. Eq. (11) is the steady state resistivity formula. It provides a direct way to obtain DC resisivity from force correlation functions. The straight forward interpretation of Eq. (12) is the force-balance: The externally applied force to keep the steady velocity is equal in magnitude but opposite in sign to the sum of the frictional and the Lorentz forces. Above analysis shows that the transverse force is not affected by the thermal reservoir under the assumption that $\eta[\omega]$ is a scalar.

So far, all of our calculations are exact. Now we will discuss how the results may change when employing the relaxation time approximation to account for the existence of thermal reservoir in comparison with the exact calculations. Without the thermal reservoir, the velocity-velocity correlation is given by

$$\mathcal{U}[\omega] = (im w - i\sigma_y B)^{-1} m\mathcal{U}(0). \tag{13}$$

Then we switch on the thermal reservoir to allow the relaxation process to happen. We use a relaxation time approximation by the standard rule, $iw \rightarrow iw + \eta[\omega]/m$. Substituting it into Eq. (13), we have found the velocity-velocity correlation under the relaxation time approximation is given by

$$\mathcal{U}[\omega] = (im\omega + \eta[\omega] - i\sigma_y B)^{-1} k_B T,$$

which is exactly the same as the one obtained by above rigorous calculation. We conclude that the relaxation time approximation can be a valid one for velocity-velocity correlations when used in a conductivity formula.

Next we evaluate the force-force correlation by the relaxation time approximation. Without thermal reservoir, the random force correlation is zero, that is, $R(t) = 0$. If we switch on the thermal reservoir by using a relaxation time approximation $iw \rightarrow i\omega + \eta[\omega]/m$, the random force correlation is still incorrectly set to zero, and cannot be made finite. The total force correlation without thermal reservoir is

$$\begin{aligned} \mathcal{F}_{ij}[w] = {} & (iB\sigma_y + im w + (m\omega)^2 \\ & (im w - i\sigma_y B)^{-1})_{ik} m\mathcal{U}_{kj}(0). \end{aligned} \tag{14}$$

When we switch on the thermal reservoir by using a relaxation time approximation $i\omega \rightarrow i\omega + \eta[\omega]/m$ in Eq. (14), we have

$$\mathcal{F}[\omega] = (iB\sigma_y + im\omega + \eta[\omega] - (im\omega + \eta[\omega])^2 \\ (im\omega + \eta[\omega] - i\sigma_y B)^{-1})k_B T.$$

This is a rather complicated expression. We can simplify it in the limit $\omega << \eta[\omega]$, or $\omega\tau << 1$, $\tau = m/\eta[0]$ is a relaxation time:

$$\mathcal{F}[0] = \frac{B}{1+(\omega_0\tau)^2}\left(\omega_0\tau + i\sigma_y(\omega_0\tau)^2\right) k_B T, \tag{15}$$

with $\omega_0 = B/m$. Let us use the resistivity formula Eq. (11) to calculate the external force needed to keep the particle moving with a given velocity. With $\mathcal{R}[\omega] = 0$ and $\mathcal{F}[0]$ given by Eq. (15), the external force is

$$< \mathbf{K}[0] >= -\frac{\omega_0\tau}{1+(\omega_0\tau)^2}(B\overline{\mathbf{u}}[0] + \omega_0\tau\overline{\mathbf{u}}[0] \times B). \tag{16}$$

These results have no connection at all to the rigorous results shown in Eq. (12). Evidently the relaxation time approximation cannot be valid in such a calculation.

However, Eq. (16) is familiar to us. With merely a re-definition of constants $B = k\rho$ with κ the circulation and ρ the superfluid density, and ω_0 as the the core level spacing, this force becomes exactly the same as the one appeared in the derivation of vortex dynamics using the relaxation time approximation [3-7], where the average is made for all other degrees of freedoms except those of the vortex. The literal interpretation of such results is that the Lorentz force on a moving particle is canceled by the effect of thermal reservoir. It also shows a different friction [13]. The similarity of structure between Eq. (16) and those obtained in vortex dynamics with relaxation time approximation suggests that they may have the same source of error. The calculation by Šimánek [4] has explicitly used the relaxation time approximation in the force-force correlation function. Although several other publications [5,6] have not explicitly specified their methods as force-force correlation function calculations, their final formula are the same as Ref. [4].

There are also exceptions. In Ref. [7], Stone made an attempt to use Boltzmann equation to solve the forces on a moving vortex. When we check Eq. (5.8) in Ref. [7], we find that if we substitute $< \mathbf{k} >=< \mathbf{k} >_0$ into this equation, where $< \mathbf{k} >_0$ is the equilibrium value of $< \mathbf{k} >$ in the frame where vortex is stationary, the equation is not satisfied. However, $< \mathbf{k} >$ should relax to $< \mathbf{k} >_0$ in such a case because there is no extra applied pinning force and the vortex is stationary in this frame. There are also additional problems. The equation itself is not Galilean invariant. Nonetheless, its solution has been transformed into lab frame using Galilean invariance in order to obtain force on a moving vortex.

Before we conclude, we return to fluctuation-dissipation theorems and Berry's phase. The type of fluctuating forces we use in stochastic process will not generate any addition transverse force in low frequency limit because of their vanishing off-diagonal correlation. However, even though they are widely used, we may still question the validation of these fluctuating forces. Our understanding is that when we separate a fluctuating force from what we leave to be systemic, we generally choose to assign the average effect to the later. Thus the fluctuating force no longer has a zero frequency component of off-diagonal correlation. The Berry's phase is more powerful in this respect. If indeed we have left an off-diagonal correlation with non-zero mean to 'random force', the zero frequency component will be included if we evaluate Berry's phase. Finally, we point out that when correctly evaluated, the core state transitions reproduces the results of Berry's phase calculation at zero temperature with impurities included, and the friction arises naturally [8,14].

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References

[1] P. Ao and D. J. Thouless, Phys. Rev. Lett. 70, 2158 (1993).
[2] D. J. Thouless, P. Ao, and Q. Niu, Phys. Rev. Lett. 76, 3758 (1996).
[3] G. E. Volovik, Sov. Phys. JETP 77, 435 (1993).
[4] E. Simanek, Phys. Rev. B52, 10336 (1995).
[5] N. B. Kopnin, Physica **B210**, 267 (1995).
[6] A. van Otterlo, M. Feigel'man, V. Geshkenbein, and G. Blatter, Phys. Rev. Lett. 75, 3736 (1995).
[7] M. Stone, Phys. Rev. B54, 13222 (1996).
[8] P. Ao and X.-M. Zhu, Physica **C282-287**, 367 (1997).
[9] R. Kubo, M. Toda, and N. Hashitsume, Statistical Physics II, 2nd ed., Springer-Verlag, Berlin, 1985.
[10] L. E. Ballentine and W. J. Heaney, J. Phys. C7, 1985 (1974).
[11] M. Huberman and G. V. Chester, Adv. Phys. 24, 489 (1975).
[12] R. S. Fishman and G. D. Mahan, Phys. Rev. **B39, 2990 (1989).**
[13] The vortex friction obtained with the relaxation time approximation sometimes carries a wrong sign. In Eq. (11) of Ref. [6], $K_c^+(\tau-\tau') > 0$. However, the friction is proportional to the coefficient of $|\mathbf{R}(\tau)-\mathbf{R}(\tau')|^2$, which differs in sign from $K_c^+(\tau-\tau') > 0$.
[14] P. Ao and X.-M. Zhu, to be published.