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Generalized Lorentz Transformations for Linearly Accelerated Frames with Limiting Four-Dimensional Symmetry

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Based on the principle of limiting four-dimensional symmetry, we discuss coordinate transformations for constant-linear-acceleration (CLA) frames. We derive a new "Wu transformation" which reduces to the Lorentz transformation in the limit of zero acceleration. The time for an accelerated frame can be realized by "computerized clocks". A 'CLA coordinate' (w, x, y, z) is preferred for the accelerated transformation and has as much physical meaning for an accelerated frame F as (w_I, x_I, y_I, z_I) for an inertial frame F_I. Furthermore, constant-linear-acceleration α must be constant increase of a particle's "energy" per unit length, in consistent with what has been realized in high energy linear accelerators. Some experimental implications are discussed.

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I. Introduction

Classical physics using Galilean transformations has been satisfactorily improved in modern physics by the use of Lorentz transformations. Yet, the corresponding classical physics in constant-linear-acceleration (CLA) frames has not been equally improved in modern physics. This is due to the lack of a generalized Lorentz transformation for CLA frames. In 1943, Moeller obtained a transformation for an inertial frame F_I and a uniformly accelerated frame F, moving along the x-axis. This was based on (a) Einstein's vacuum equation $R_{ij} = 0$ and (b) time-independent metric tensors, $ds^2 = g_{00}(x)c^2dt^2 - dx^2 - dy^2 - dz^2$. He obtained $g_{00}(x) = (1 + gx)^2$ and a transformation for uniformly accelerated frames [1]. The reason for using $R_{ik} = 0$ is suggested by a heuristic view that the 'inertial force' of accelerated frames and the 'gravitational force' may be considered as being "unified" by Einstein's equation. Nevertheless, these two forces satisfy quite different "boundary conditions": Namely, in contrast to the gravitational force, the inertial force does not vanish at spatial infinity and the transformations for accelerated frames should reduce to the Lorentz transformation when inertial forces vanish. In 1972, Wu and Lee derived the same transformation based on a "*kinematic approach*" without using Einstein's equation for gravity [1]. But, Moeller's transformations cannot be smoothly connected to the Lorentz

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transformations in the limit of zero acceleration. This is due to the stringent assumption that the metric tensor g_{00} is time-independent.

In this paper, we follow the kinematic approach and obtain a satisfactory CLA transformation by postulating a new and natural principle of "limiting four-dimensional symmetry" [2]: Any accelerated transformation of coordinates must reduce to the form with 4-dimensional symmetry in the limit of zero acceleration. We show that the set of transformations for the CLA frames forms a new group, which is termed the "Wu group". The "Wu transformation" is a natural and simple generalization of the Lorentz transformation and the Galilean transformation with constant acceleration. The limiting four-dimensional symmetry principle contains more definite and satisfactory physical results than the equation $R_{ik}= 0$, as far as CLA transformations are concerned. In the gravitational approach, Einstein's covariant equation holds for any coordinate. However, the Lorentz transformation prefers the Cartesian coordinate. Therefore, the natural assumption of the smooth connection between a linearly accelerated frame and an inertial frame dictates that a CLA coordinate is preferred for CLA transformations. This is an important difference between kinematic and gravitational approaches.

In our discussions, a CLA frame $F(w,x,y,z)$ with the usual definition $w = ct$ is introduced. But we know that the constant speed of light c has no operational definition in any CLA frame. Fortunately, the physical results in the paper are actually independent of the definition $w = ct$. In previous papers, we have shown that the logically simplest theory of relativity, called "taiji relativity," can be formulated solely on *the basis of the first postulate of relativity, without* making any second *postulate* concerning *the speed of light* [3]. The first postulate of relativity states that the laws of physics have the same form in all inertial frames. We are able to formulate a 4-dimensional physical theory with the coordinate $x_I^\mu = (w_I,x_I,y_I,z_I)$ for an inertial frame F_I, where w_I is the 'taiji-time' with the dimension of length. The absence of the second postulate forbids one to express the taiji-time w_I in terms of the usual time t_I (measured in seconds) and velocity because they cannot be defined for all inertial frames in taiji relativity. Nevertheless, the taiji-time w_I can be directly used as the evolution variable. Furthermore, the taiji-time with the unit **of,** say, centimeters, can be physically realized by computerized clocks. Also, the invariant law for the propagation of light, $ds^2 = dw_I^2 - dr_I^2 = 0$, implies the 'taiji-speed' of a light signal to be *dimensionless* and has the universal value, $\beta_L = |dr_I/dw_I| = 1$, for all inertial frames. A careful examination shows that taiji relativity is consistent with all previous experiments [3]. Indeed, one can simply consider w in a CLA frame as the evolution variable for a physical system. One can have a grid of computerized clocks in a CLA frame. These clocks can be synchronized without relying on the constant speed of light signals and will automatically read taiji-time w_I in the limit of zero acceleration.

II. Coordinate transformations for linearly accelerated frames

Suppose we have an inertial frame $F_I(w_I,x_I,y_I,z_I), w_I = ct_I$, and a CLA frame $F(w,x,y,z)$ moving with a constant acceleration a along the x-axis. Based on the preceding discussions, it is natural to assume that ds^2 takes the form

$$\begin{aligned} ds^2 &= c^2 dt_I^2 - dx_I^2 - dy_I^2 - dz_I^2 \\ &= g_{00}(x,w)dw^2 + g_{11}(x)dx^2 + g_{22}(x)dy^2 \;\mathbf{t}\; g_{33}(x)dz^2, \end{aligned} \tag{2.1}$$

where t_I is shown by the Einstein clocks in the inertial frame F_I. As usual, one may define w = *ct,* where the realization of t through a grid of computerized clocks will be discussed later in sec. 6. Although $R_{ik} = 0$ holds for arbitrary coordinates, we postulate the metric (2.1) so that ds^2 and the resultant transformations are compatible with both Einstein's vacuum equation $R_{ik} = 0$ and the new "boundary conditions" of limiting four-dimensional symmetry. Since the CLA frame F moves along the z-axis, we look for axial symmetric solutions with

$$g_{22} = g_{33} = -Y^2(x), \tag{2.2}$$

and all metric tensors are functions of x, except that g_{00} may be a function of x and w. This property of $g_{00}(x,w)$ is crucial for the new CLA transformation.

II-l. Gravitational approach

Let us first consider the conventional "gravitational approach" based on Einstein's equation to obtain uniformly accelerated transformations. Based on (2.1) and (2.2). We can calculate Christoffel symbols $G^i_{jk} = g^{im}(\partial_k g_{mj} + \partial_j g_{mk} - \partial_m g_{jk})$ and the Ricci tensor $R_{ik} = \partial_m G^m_{ik} + \partial_k G^m_{im} + G^n_{ik} G^m_{nm} - G^m_{in} G^n_{km}$. The equations $R_{ii} = 0$, $i = 0, 1, 2$, lead to

$$\partial_x^2 W - \partial_x W(\partial_x X/X - 2\partial_x Y/Y) = 0, \tag{2.3}$$

$$\partial_x^2 W/W + 2\partial_x^2 Y/Y - (\partial_x X/X)[2\partial_x Y/Y + \partial_x W/W] = 0, \tag{2.4}$$

$$\partial_x^2 Y/Y - \partial_x Y \partial_x X/(YX) + (\partial_x Y/Y)^2 + \partial_x Y \partial_x W/(YW) = 0, \tag{2.5}$$

respectively, where Y is given by (2.2) and

$$W^2 = g_{00}(x,w), \quad X^2 = -g_{11}(x); \quad \mathrm{w} = W_1(x)W_2(w). \tag{2.6}$$

We note that $R_{33} = 0$ gives the same equation as (2.5) and other components of R_{ik} vanish identically.

If $\partial_x Y \neq 0$, equations (2.3)–(2.5) leads to an exact solution

$$W_1 = f_3/(f_1 x \;\mathrm{t}\; f_o)^{1/2}, \quad X = f_2(f_1 x \;\mathrm{t}\; fo)^{1/2}, \quad Y = f_1 x + f_o, \tag{2.7}$$

where f's are constants. We stress that $W_2(w)$ in (2.6) is arbitrary because it cannot be determined by Einstein's equation. Physically, one expects that the metric tensor g_{22} should satisfies $-g_{22} = Y^2 = 1$ rather than $Y^2 = (f_1 x + f_o)^2$ since there is no motion along the y-axis. Furthermore, the accelerated transformation based on this solution cannot be smoothly connected to the Lorentz transformation (i.e., it does not satisfy the "limiting four-dimensional symmetry" or the integrability conditions in (2.15) below.) Therefore, the solution **(2.7)** is not physically meaningful.

Let us concentrate on the case $\partial_x Y = 0$ and $\partial_x W \neq 0$. We have the solution Y = 1, which satisfies the boundary conditions $g_{22}(0) = g_{33}(0) = -1$ at the origin. From Eqs. (2.3) and (2.8), we deduce a general relation between $W_1(x)$ and X(z):

$$dW_1(x)/dx = fX(x), \tag{2.8}$$

where f is a constant of integration. Furthermore, the time-dependent part of g_{00}, i.e., $W_2(w)$, still cannot be determined by Einstein's equation, just like the previous case $\partial_x Y \neq 0$. Thus we have seen that Einstein's covariant equation by itself does not lead to a specific form for X(z), $W_1(x)$ and $W_2(w)$. (If one further postulates the limiting 4-dimensional symmetry, then $W_2(w)$ can be determined and one can obtain a CLA transformation based on $W_2(w)$, Eq. (2.8) and a suitable initial condition. But if one postulate the limiting 4-dimensional symmetry, then one can obtain the same CLA transformation without using Einstein's equation at all, as we shall see below in sec. 11-2.)

Moeller made two additional postulates, $X(x) = W_2(w) = 1$, and obtained [1]

$$ds^2 = g_{00}(x)c^2dt^2 - dx^2 - dy^2 - dz^2, \qquad g_{00}(x) = (1\ t\ gx)^2. \tag{2.9}$$

This leads to Moeller's transformation involving only one parameter g. When the acceleration g approaches zero, it does not reduce to a Lorentz transformation with a constant velocity. This is due to the lack of a velocity parameter in (2.9) which is intimately related to his stringent assumption that the metric tensor g_{00} is time-independent, i.e., $W_2(w) = 1$, as we shall see below.

11-2. **Kinematic approach**

Next, let us consider a new "kinematic approach" based on the principle of limiting 4-dimensional symmetry. Since F accelerates along the x-axis, the perpendicular coordinates y and z should not appear in the metric tensors g_{ik} and $g_{22} = g_{33} = -1$ in (2.1). The length of a measuring rod along the x-axis or the component g_{11} should not depend on time or w because the acceleration is characterized by a constant.

Based on the limiting four-dimensional symmetry, the invariant interval (2.1) with $g_{22} = g_{33} = -1$ leads to the following differential form of transformation between F_I and F,

$$d(ct_I) = \gamma(Wdw + \beta Xdx), \quad dx_I = \gamma(Xdx + \beta Wdw), dy_I = dy, \; dz_I = dz; \tag{2.10}$$

where

$$\beta = \alpha w + \beta_o, \quad \mathbf{y(w) = (1 - \beta^2)^{-1/2} \equiv 7;} \tag{2.11}$$

$$[g_{00}(w,x)]^{1/2} = G(w)Z(x) \equiv W(w,x) > 0; \tag{2.12}$$

$$(-g_{11})^{1/2} = X(x) > 0, \tag{2.13}$$

The CLA transformation (2.10) is characterized by two parameters: acceleration α and initial velocity β_0. In order to satisfy the limiting four-dimensional symmetry, we must have

$$g_{00} \rightarrow 1 \quad \text{and} \quad g_{11} \rightarrow -1 \quad \text{as } \alpha \rightarrow 0. \tag{2.14}$$

Also, the coefficients of *dx* and *dw* in (2.10) must satisfy the integrability conditions

$$\partial(\gamma W)/\partial x = \partial(\gamma\beta X)/\partial w, \qquad \partial(\gamma X)/\partial w = \partial(\gamma\beta W)/\partial x, \tag{2.15}$$

so that we have a finite coordinate transformation. It follows from (2.12) and (2.15) that

$$G(w) = \alpha\gamma^2(w)/f, \tag{2.16}$$

$$dZ(x)/dx = fX(x), \tag{2.17}$$

where the constant f is coming from separation of variables w and x. Using (2.16) and (2.17), we can integrate (2.10) to obtain the finite transformation between F_I and F,

$$ct_I = \gamma\beta Z(x)/\alpha - \beta_0/\alpha\gamma_0, \quad x_I = \gamma Z(x)/\alpha - 1/\alpha\gamma_0, \quad y_I = y, \quad z_I = z; \tag{2.18}$$

$$Z(x) \sim (\gamma_0^{-2} + \alpha x + \cdots) \quad \text{for small } \alpha; \quad f = \alpha; \quad \gamma_0 = (1-\beta_0^2)^{-1/2}. \tag{2.19}$$

Note that the constants of integration in (2.18) and the relations in (2.19) are all determined by the limiting four-dimensional symmetry and a boundary condition at the origin, $Z(0) < \infty$. In order to determined the precise form for the function Z(x), we observe that the Lorentz transformation reduces to the identity transformation, $\mathbf{r}'=\mathbf{r}$, when time and velocity vanish, $t = 0$ and $V = 0$. Thus, it is natural to impose the same initial condition to the accelerated transformation (2.18): Namely, when time $t_I = 0$ and velocity $\beta_0 = 0$, transformation (2.18) reduces to the identity transformation,

$$\mathbf{r} = \mathbf{r}_I, \tag{2.20}$$

for all values of acceleration α. It follows from (2.18)–(2.20) that Z(x) must be

$$Z(x) = (\gamma_0^{-2} + ox), \qquad \text{or} \qquad X = 1 = (-g_{11})^{1/2}, \tag{2.21}$$

Thus, *the metric tensor in* (2.1) *is now completely determined by the limiting four-dimensional symmetry and the initial condition (2.20).*

From Eqs. (2.18) and (2.21), we obtain a definite coordinate transformation between the inertial frame F_I and the accelerated frame F:

$$\begin{aligned} w_I &= \gamma\beta(x + 1/\alpha\gamma_0^2) - \beta_0/\alpha\gamma_0, \\ x_I &= 7(x \quad t \quad 1/\alpha\gamma_0^2) - 1/\alpha\gamma_0, \; y_I = y, \; z_I = z; \; w_I = ct_I, \end{aligned} \tag{2.22}$$

where the time $t_I = w_I/c$ is shown by the conventional Einstein clocks in the inertial frame F_I. We shall call the result (2.22) the "Wu transformation" [4]. One can verify that the transformation (2.22) with $w = ct$ includes the Lorentz transformation, $w_I = \gamma_0(x + \beta_0 w), x_I = \gamma_0(x + \beta w)$, as a special case, $\alpha \rightarrow 0$. In this sense, it satisfies the limiting four-dimensional symmetry, $s^2 = c^2t_I^2 - \mathbf{r}_I^2 = w^2 - \mathbf{r}^2$ as $\alpha \rightarrow 0$. The inverse Wu transformation of (2.22) can be deduced:

$$\begin{aligned} w &= (ct_I + \beta_0/\alpha\gamma_0)/[\alpha(x_I + 1/\alpha\gamma_0)] - \beta_0/\alpha, \\ x &= [(x_I + 1/\alpha\gamma_0)^2 - (ct_I + \beta_0/\alpha\gamma_0)^2]^{1/2} - 1/\alpha\gamma_o^2. \quad y = y_I, \quad z = z_I. \end{aligned} \tag{2.23}$$

We may remark that when $\beta_0 \rightarrow 0$, one can verify that (2.23) leads to the accelerated Galilean transformation, $x \approx x_I - c^2\alpha t_I^2/2$. Thus, the acceleration α can be approximately related to a constant acceleration g in Newtonian mechanics, $\alpha \approx g/c^2$. The transformation for the covariant differential operators $(\partial/\partial w,\ \partial/\partial \mathbf{r})$ can be deduced from (2.23):

$$\begin{aligned} \partial/\partial w_I &= \gamma(w^{-1}\partial/\partial w - \mathrm{pa/ax>}, \\ \partial/\partial x_I &= \gamma(\partial/\partial x - \beta W^{-1}\partial/\partial w), \quad \partial/\partial y_I = \partial/\partial y, \partial/\partial z_I = \partial/\partial z. \end{aligned} \tag{2.24}$$

These relations will be useful for wave equations in quantum mechanics.

III. The Wu group of constant-linear-acceleration transformations

To show the group properties of the Wu transformation (2.22) for CLA frames, we must first find the coordinate transformations between two CLA frames F and F' · Let us consider another frame F' accelerating with a velocity $\beta' = \mathrm{Q}'w'$ with respect to F_I. For simplicity, we shall ignore trivial y' and z' axes and set all initial velocities to zero in the following discussions. *However' these group properties can be* shown *to be true also for non-zero initial velocities.* Similar to (2.22), we can write down the Wu transformation between the inertial frame F_I and the accelerated frame F',

$$\begin{aligned} w_I &= \gamma' Z' w', \quad x_I = \gamma' Z'/\alpha' - 1/\alpha'; \\ Z' &= 1 + \alpha' x', \quad \beta' = \alpha' w', \quad 7' = (1 - \beta'^2)^{-1/2}. \end{aligned} \tag{3.1}$$

Using (2.14) with $\beta_0 = y = z = 0$ and (3.1), we deduce the Wu transformation between F' and F:

$$\begin{aligned} \mathrm{w} &= \gamma' Z' w'/\{\alpha[Z'\gamma'/\alpha' + 1/\alpha - /\alpha']\}, \\ \mathrm{x} &= \{[\gamma' Z'/\alpha' + 1/\alpha - 1/\alpha']^2 - \gamma'^2 Z'^2 w'^2\}^{1/2} - 1/\alpha. \end{aligned} \tag{3.2}$$

In order to see the group property of the Wu transformation (3.2) for two accelerated frames, we need to consider a third accelerated frame F'' with a velocity $\beta'' = \alpha'' w''$. In analogy to (3.2), we can write down the transformation between F'' and F,

$$\begin{aligned} \mathrm{w} &= \gamma'' Z'' w''/\{\alpha[Z''\gamma''/\alpha'' + 1/\alpha - /\alpha'']\}, \\ x &= \{[\gamma'' Z''/\alpha'' + 1/\alpha - 1/\alpha'']^2 - \gamma''^2 Z''^2 w''^2\}^{1/2} - 1/\alpha; \\ Z'' &= 1 + \alpha'' x'', \quad \gamma'' = (1 - \beta''^2)^{-1/2}. \end{aligned} \tag{3.3}$$

Based on (3.3) and the inverse of (3.2), the transformation between F' and F'' is

$$\begin{aligned} \mathrm{w}' &= \gamma'' Z'' w''/\{\alpha'[Z''\gamma''/\alpha'' + 1/\alpha' - /\alpha'']\}, \\ x' &= \{[\gamma'' Z''/\alpha'' + 1/\alpha' - 1/\alpha'']^2 - \gamma''^2 Z''^2 w''^2\}^{1/2} - 1/\alpha'. \end{aligned} \tag{3.4}$$

which has the same form as that of (3.2). Using (3.1) and (3.4), $s^2 = (ct_I)^2 - x_I^2$ can be expressed in terms of the coordinate variables in F' and F'' as follows:

$$s^2 = (ct_I)^2 - x_I^2 = [2Z'\gamma' - Z'^2 - 1]/\alpha'^2 = [2Z''\gamma'' - Z''^2 - 1]/\alpha''^2. \tag{3.5}$$

The existence of identity transformation and the associate law can also be verified. Thus we conclude that the set of Wu transformations for accelerated frames form a group which is termed the Wu group. Since a Wu transformation (with $w = ct$ or w ' $= ct'$, etc.) reduces to a Lorentz transformation in the limit of zero acceleration, the Wu group includes the Lorentz group as a special case when all accelerations vanish.

One can generalize the Wu transformation in such a way that the velocity u $\equiv \vec{u}$ is in arbitrary and fixed direction. This implies that both the acceleration $\vec{\alpha}$ and the initial velocity β_0 must be in the same direction:

$$\mathbf{u} = \vec{\alpha} w + \vec{\beta}_0. \tag{3.6}$$

By differentiating (2.14), we obtain

$$d(ct_I) = \gamma(W dw + \beta dx), \quad dx_I = \gamma(dx + \beta W dw), dy_I = dy, dz_I = dz; \tag{3.7}$$

where

$$W = \gamma^2(\gamma_0^{-2} + \alpha x); \qquad \gamma_0 = (1 - \beta_0^2)^{-1/2}.$$

In analogy with the Lorentz transformation in an arbitrary direction, (3.7) can be generalized to the following form:

$$\begin{aligned} d\mathbf{r}_I &= dr + (\gamma - 1)(\mathbf{u}\cdot d\mathbf{r})\mathbf{u}/u^2 + [\gamma^3\gamma_0^{-2}\mathbf{u} + \alpha\gamma^3(\mathbf{u}\cdot\mathbf{r})\mathbf{u}/u]dw, \\ &= d\mathbf{r} + (\gamma - 1)(\mathbf{u}/u)d(\mathrm{u} \cdot \mathbf{r}/u) + (1/\alpha\gamma_0^2)(\mathbf{u}/u)d\gamma + (\mathbf{u}\cdot\mathbf{r})(\mathbf{u}/u^2)d(\gamma - 1), \\ dw_I &= \gamma^3[\gamma_0^{-2} + au \cdot \mathbf{r}/u]dw + \gamma\mathbf{u}\cdot \mathrm{dr}. \\ &= \gamma^3\gamma_0^{-2}dw + \gamma(\mathbf{u}\cdot\mathbf{r}/u)du(\gamma^2 - 1) - \gamma^3(\mathbf{u}\cdot\mathbf{r})udu \mathrm{\ t\ } d(\gamma\mathbf{u}\cdot \mathrm{r}), \end{aligned} \tag{3.8}$$

where $u = |\mathbf{u}|$. It is straightforward to carried out the integrations, we have

$$\begin{aligned} \mathbf{r}_I &= \mathbf{r} + (\gamma - 1)(\mathbf{u}\cdot\mathbf{r})\mathbf{u}/u^2 + \mathbf{u}[\gamma\gamma_0^{-2} - \gamma_0^{-1}]/(\alpha u), \\ w_I &= (1/\alpha\gamma_0^2)(\gamma u - \gamma_0\beta_0) \mathrm{\ t\ } \gamma\mathbf{u}\cdot\mathbf{r}. \end{aligned} \tag{3.9}$$

It can be verified that the general transformation (3.9) reduces to (2.22) if u is in the x-direction, u $= (\beta, 0, 0)$. In the zero acceleration limit, (3.9) reduces to

$$\begin{aligned} \mathbf{r}_I &= \mathbf{r} + (\gamma_o - 1)(\vec{\beta}_o\, \mathbf{r})\vec{\beta}_o/\beta_o^2 + \vec{\beta}_o\gamma_o w, \\ w_I &= w\gamma_o \mathrm{\ t\ } \gamma_o\vec{\beta}_o\cdot\mathbf{r}, \end{aligned} \tag{3.10}$$

which is the well-known form of the general Lorentz transformation.

IV. Physics in linearly accelerated frames

Within the four-dimensional symmetry framework of taiji relativity based solely on the first postulate of relativity [3], it was shown that there are only two universal and fundamental constants: $J =$ 3.5177293x 10^{-38} g·cm and $\bar{e}$ = -1.6021891 x $10^{-20}(4\pi)^{1/2}(\mathrm{g}\cdot\mathrm{cm})^{1/2}$, instead of the usual three, h, e (in esu) [5] and c. These results can be applied to the present formalism of physics in CLA frames.

Since *the speed of light in a CLA frame F is not a universal constant, we shall write* $w \equiv bt$ *and* $u \equiv dw/dt$, *where b is a variable in this section, so that one can see that the physics* in *F does not depend on* $w = ct$. The invariant action for a charged particle and the electromagnetic potential $a_\mu(x)$ in F is assumed to be

$$5' = \int(-mds - \bar{e}a_\mu dx^\mu) - (1/4)\int f_{\mu\nu}f^{\mu\nu}Wd^4x = \int Ldt - (1/4)\int f_{\mu\nu}f^{\mu\nu}Wd^4x; \qquad (4.1)$$

$$ds^2 = W^2dw^2 - dx^2 - dy^2 - dz^2, \quad (\text{-}det\ g_{\mu\nu})^{1/2} = W; \quad w = bt; \qquad (4.2)$$

$$L = -m[W^2u^2 - v_x^2 - v_y^2 - v_z^2]^{1/2} - \bar{e}(a_0u + a_iv^i), \quad u = dw/dt, \quad \mathbf{v} = d\mathbf{r}/dt; \qquad (4.3)$$

$$f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu. \quad g_{\mu\nu} = (W^2, -1, -1, -1). \qquad (4.4)$$

In the limit $\alpha \to 0$, F becomes an inertial frame and, hence, $\bar{e}$ and a, correspond to the usual charge e (in esu) and the electromagnetic potential A, by $\bar{e} =$ e/c and $a_\mu = A_\mu/c$ respectively. The canonical momentum P_i of a particle in F is given by'

$$P_i = -\partial L/\partial v^i = p_i + \bar{e}a_i; \quad i = 1,2,3; \qquad (4.5)$$

$$p_i = (-m\gamma v_x/C, -m\gamma v_y/C, -m\gamma v_z/C) = g_{ik}p^k, \quad p' \equiv mdx^i/ds; \qquad (4.6)$$

$$7 = (1 - v^2/C^2)^{-1/2}; \quad C \equiv uW, \quad v^2 = v_x^2 + v_Y^2 + v_z^2 \equiv -v_iv^i. \qquad (4.7)$$

The "Hamiltonian" $H = P_0$. with the same dimension as that of P_i, is defined *by*

$$P_0 \equiv [(\partial L/\partial v^i)v^i - L]/u = p_0 + \bar{e}a_0 = W[(P_i - \bar{e}a_i)^2 + m^2]^{1/2} + \bar{e}a_0 \equiv H; \qquad (4.8)$$

$$p_0 = m\gamma W = g_{00}p^0, \quad p^0 \equiv mdx^0/ds = mdw/ds. \qquad (4.9)$$

The transformation of the covariant momenta p_i and p_0 is given by

$$p_{I0} = \gamma(p_0/W - \beta p_1), \; p_{I1} = \gamma(p_1 - \beta p_0/W), \; p_{I2} = p_2, \; p_{I3} = p_3; \qquad (4.10)$$

where

$$p_\mu = (p_0, -\mathbf{p}), \quad \mathbf{p} = (p_x, p_y, p_z) = (p^1, p^2, p^3) = (-p_1, -p_2, -p_3).$$

Note that *(4.10)* is consistent with (2.24) because p_μ and $\partial/\partial x^\mu$ should have the same transformation property. Also, the CLA transformation of the covariant vector *dx,* is the same as (4.10) because dx, = $g_{\mu\nu}dx^\nu$ and $dx_{I\mu} = \eta_{\mu\nu}dx_I^\nu$, where $\eta_{\mu\nu} = (1,-1,-1,-1)$. The invariant relation $g^{\mu\nu}p^\nu p^\nu = m^2$ implies

$$\begin{aligned} &g^{\mu\nu}(P_\mu - \bar{e}a_\mu)(P_\nu - \bar{e}a_\nu)\\ &= W^{-2}(P_0 - \bar{e}a_0)^2 - (P_1 - \bar{e}a_1)^2 - (P_2 - \bar{e}a_2)^2 - (P3 - \bar{e}a_3)^2 = m^2, \end{aligned} \qquad (4.11)$$

where we have used (4.5), (4.6), (4.8), (4.9), $g^{\mu\nu} = (W^{-2}, -1, -1, -1)$ and $P^{\mu} = g^{\mu\nu}P_{\nu}$. *This* equation suggests that the generalized Dirac equation for the accelerated frame F should have the form

$$[\gamma^{*\mu}(x)(P_{\mu} - \bar{e}a_{\mu}) - m]\Psi = 0, \qquad P_{\mu} = i\mathbf{J}\partial/\partial x^{\mu}. \tag{4.12}$$

If one wishes, one can relate y*(s) in (4.12) to constant Dirac matrices $\overline{\gamma}^{\mu}$ by the relation $\gamma^{*\mu}(x) = e^{\mu}_{\nu|}(x)\overline{\gamma}^{\nu}$, where $e^{\mu}_{\nu|}(x)$ is a tetrad.

V. Experimental Implications and Discussions

The result (4.10) and the new transformation (2.22) can be experimentally tested by measuring Doppler shift of wavelength of light emitted from a CLA source. From Eq. (4.10) one obtains the transformation of the covariant wave 4-vector $k_{\mu} = p_{\mu}/J$ between an inertial frame F_I and a CLA frame F. Note that Jk_{I0} and Jk_0 are moving masses of the same photon measured from F_I and F respectively. Suppose the radiation source is at rest at the origin of the F frame, $\mathbf{r} = 0$, and $k_{\mu} = (k_0, -k_1, 0, 0)$, where $k_0 = k_0$ (rest) and $k_1 = k_1$ (rest). Experimentally, it is difficult to measure k_0 (rest) and k_1 (rest) in the CLA frame. Thus we have to express them in terms of quantities measured in the inertial frame (or laboratory) F_I. Using (4.10), the relation [6] $k_0(\text{rest}) = k_{I0}(\text{rest})$ and Z(0) = γ_o^{-2}, we obtain the shifts of k_{I0} (related to photon's "moving mass" or atomic mass level [3]) for waves emitted from a CLA source,

$$k_{I0} = k_{I0}(\text{rest})[\gamma_o^2(1 + \beta)/\gamma], \quad k_1 = -k_x; \quad \beta = \alpha w + \beta_o, \tag{5.1}$$

where (rest) denotes the source being at rest in F_I. A similar relation can be obtained for the wavelength. Such new effects predicted by the Wu transformation for waves emitted from a CLA source may be termed Wu-Doppler effect. Note that Moeller's transformation will lead to a result different from (5.1) because t in (2.9) and (2.22) with w = ct [or (5.2) below] must have the same physical interpretation. Such a difference can be tested by measuring the Wu-Doppler effect (5.1) in the laboratory frame F_I by using the method of Ives-Stilwell [7].

We stress that the Wu transformation (2.22) does not depend on a specific relation between w and t. Suppose one assumes w = ct in (2.22). The time t,

$$t = [t_I + \beta_o/(c\alpha\gamma_o)]/[\alpha(x_I + 1/\alpha\gamma_o)] - \beta_o/(c\alpha), \tag{5.2}$$

in the accelerated frame F can be physically realized by the "computerized clocks" [3]: By basing "computerized clocks" on a computer chip, one could program any clock in F to obtain a time reading t_I from the nearest Einstein clock in F_I and, based on c, β_o, α and its F_I frame position x_I, compute the time t it should display according to (5.2). This is a general method for synchronization of computerized clocks in a reference frame, without relying on the constant speed of light. If one compares the *rate* of *ticking* of the Einstein and the computerized clocks at a fixed position x_I, one has $(\partial t/\partial t_I)_{x_I} = 1/[\alpha x_I + 1/\gamma_0]$. *This can be physically realized* because *the* reading and *the rate of ticking of a clock are adjustable. The* "computerized *clocks" in F will* automatically *becomes Einstein clocks* as *the* acceleration ***approaches zero,*** $\alpha \rightarrow 0$. In this sense, these sophisticated computerized clocks are generalized Einstein clocks for both inertial and non-inertial frames. We note that the choice of w $= ct$ in (2.22) to synchronize computerized clocks in ***F*** does not imply that the speed of light is a constant c in the accelerated frame. For the general case w = bt, where t is defined by an arbitrarily preassigned function $t(x_I, t_I)$, the time t can also be physically realized by the "computerized clocks". It appears that all these different times $t(x_I, t_I)$ are equally physical, in principle, for describing physical phenomena, as discussed in [3]. However, from fundamental laws of physics such as (4.11) and (4.12), we can see that the real evolution variable is w rather than t. Of course, we can also make these computerized clocks to read w directly.

The boundary condition Z(0) $< \infty$, which leads to $f = \alpha$ in (2.19), is imposed for simplicity. It is not necessary: If Z(0) $< \infty$ is not imposed, then one has $f \neq \alpha$ in general. However, G(w) and $Z(x)$involve the factors $1/f$ and f respectively, so that $W(w,x) = G(w)Z(x)$ and the resultant physics do not depend on f.

The equivalence of the effects of a gravitational field and those of an observer's acceleration played an essential role at the birth of general relativity. Nevertheless, some authors suggested that it be buried with appropriate honors because it is false [9]. It is the equivalence of gravitational and inertial mass which is precise and necessary for general relativity.

What is the operational meaning of the constant acceleration α? We show that the constant acceleration α of a particle is directly and uniquely related the change of its energy per unit length as measured in an inertial frame F_I[10]:

$$(dp_{I0}/dx_I)_x = m\alpha/Z(x), \qquad x = \text{fixed}, \tag{5.3}$$

where we have used the differential transformation (2.10) and the momentum transformation (4.10) with p_μ = Jk,. We stress that, within the framework of the four-dimensional symmetry, the concept of uniform acceleration of a particle can only be defined in the sense of (5.3), i.e., constant change of a particle's "energy" p_{I0} per unit length, as measured in an inertial frame F_I. It is gratifying to see that this is precisely what has been used in high energy laboratory. Other definition of acceleration such as the change of velocity per unit time is only an approximation for small velocities and is, strictly speaking, incompatible with the 4-dimensional symmetry.

Our results suggest that the kinematic approach first discussed by Wu and Lee [1] is more fruitful than the conventional gravitational approach, provided the limiting 4-dimensional symmetry is postulated.

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time-dilatation and (iii) a time-independent g_{00} for CLA coordinates, and derived the same transformation.

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[4] Ta-You Wu, *Theoretical Physics,* vol. *4: Theory of Relativity* (in Chinese, Lian Jing Publishing Co., 1978) pp. 172-175; see also ref. 1. Roughly speaking, Wu explored local relation between "accelerated transformation" and Lorentz transformation; while we consider their global relations based on the limiting four-dimensional symmetry principle.

[5] Note that the universal constant $\bar{e}$ is the charge measured in the electromagnetic unit (emu) rather than in the electrostatic unit (esu).

[6] This equality is only approximate because there is no relativity or equivalence between F and F_I. Nevertheless, it turns out to be an extremely good approximation because atomic structure is very stable against constant-linear-acceleration. This is basically related to the metric tensor g_{00} and the smallness of atomic sizes, $\sim 10^{-8}$ cm.

[7] When we define $w = ct$, the speed of light measured in the CLA frame F is C $\equiv dr/dt = cW(t, x)$ because the propagation of light is described by equation (2.1), $ds = 0$. Note that C is anisotropic and depends on space and time in general. We stress that one can also set w = bt_I in (2.22) and (2.23) *without upsetting its Wu group property*. Thus, we have a common time, $t = t_I$, for all frames and, hence, the speed of light measured in F by using such a common time will be C = $c\gamma(1-\beta)$, if $dx_I/dt_I = +c$. For discussions of common time within the 4-dimensional framework, see J. P. Hsu, Phys. Lett. **A97,** 137 (1983); Nuovo Cimento B74, *67* (1983); J. P. Hsu and C. Whan, Phys. Rev. **A38**, 2248 (1988), Appendix.

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[10] For an object at rest in F_I, *we* have $(dp_0/dx)_{x_I} = m\alpha(\gamma^{-1}-1)/(\gamma Z^2) \neq (dp_{I0}/dx_I)_x$. It also shows the lack of symmetry between a CLA frame F and an inertial frame F_I.