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Recent Developments in the Theory of the Quantum Hall Effect

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The past few years have produced major advances in our understanding of the quantum Hall effects – quantized and unquantized. Theories based on a mathematical transformation, where the electrons are replaced by a set of fermions interacting with a Chern-Simons gauge field, have been useful for explaining and predicting observations at even-denominator filling fractions where quantized Hall plateaus are not observed, as well as for giving new insight into the most prominent fractional quantized Hall states at odd-denominator fractions. Other theoretical approaches have led to important advances in our understanding of edge-excitations for systems in a fractional quantized Hall state, of phases and phase transitions in bilayer systems, of tunneling phenomena in the quantum Hall regimes, and of disorder-induced transitions between "neighboring" quantum Hall plateaus. Some highlights of these developments are reviewed.

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I. INTRODUCTION

During the past few years, there has been very substantial progress, both theoretical and experimental, in our understanding of the behavior of electrons in a partially-filled Landau level. Some of this work has focused on the nature of the quantized Hall states, in which the Hall resistance ρ_{xy} is locked to a rational multiple of the unit h/e^2, while the longitudinal resistivity ρ_{xx} is found to vanish in the limit of low temperatures. Other work, at least equally interesting, has focused on Landau-level filling-fractions f where the quantized Hall effect is not observed. A number of remarkable phenomena have been observed under these conditions, and we find that the effects of electron-electron interactions in a two-dimensional electron system in a strong magnetic field, can be quite subtle, even when the Hall conductance is not quantized. It seems appropriate to denote these phenomena, collectively, as the unquantized quantum Hall effect (UQHE).

In the following sections, I will try to outline developments in several selected areas that I have found to be particularly exciting or challenging. The references cited are representative examples intended to give the reader an entrance into the literature of these subjects. I have not been able to compile anything approaching a complete listing of the significant contributions, nor do I claim to have selected the most important references in each category. I apologize in advance for the references omitted.

II. FERMION CHERN-SIMONS THEORY

The unquantized Hall effect has been especially interesting for very high mobility samples at or near an even-denominator fraction such as $f = 1/2$. The theoretical approach which has proved most useful in this case is based on a mathematical transformation to a system of fermions ("composite fermions") interacting with a fictitious gauge field of the Chern-Simons type [1-3]. (Composite fermions were originally introduced by Jain to explain the most prominent fractional quantized Hall states, at fractions of the form $f = p/(2p+1)$, where p is an integer [4-6]. Theories based on a transformation to a system of *bosons* interacting with a Chern Simons field had also been used previously to discuss the quantized Hall effect [6,7].)

A key result has been the realization that at $f = 1/2$, and at various other even denominator filling fractions, there are fermionic quasiparticles which move in straight lines, and behave in many ways like quasiparticles of a Fermi liquid in zero magnetic field [1]. For magnetic fields which deviate by a small amount AB from the magnetic field $B_{1/2}$ which corresponds to $f = 1/2$, the quasiparticles move in circles whose radius R_c^* is equal to the cyclotron radius for a particle in the *effective field* AB. This has important experimental consequences, which have been beautifully demonstrated by several experiments during the past year [8-10]. Particularly impressive are surface acoustic wave experiments which show a resonance feature when the effective cyclotron diameter $2R_c^*$ coincides with a theoretically-predicted multiple of the acoustic wavelength [8]. Other experiments have seen effects in transport properties when $2R_c^*$ is related to geometric features of an imposed lithographic structure [9,10]. Taken together, these experiments confirm that fermionic quasiparticles exist and follow the prescribed trajectories at least over distances of several microns, a hundred times larger than the electron-electron separation or the true cyclotron diameter of an electron in the lowest Landau level.

An important open question is whether some type of modified Fermi liquid theory can hold, in principle, all the way down to zero energy and infinite length scales, at zero temperature, precisely at $f = 1/2$, in an ideal sample with no impurity scattering. If so, is there a finite effective mass for the fermions, or is the effective mass m^* singular in the limit $|E - E_F| \to 0$? The original analysis of Ref. 1 suggested that m^* should diverge as $\ln|E - E_F|$, if the electron-electron repulsion has the Coulomb form, $\propto 1/r$ for large separations r. More recent investigations do not necessarily agree with this conclusion, however [11]. It is also possible that different definitions could lead to different results for the effective mass, One choice, which is at least well defined in principle, is to use the energy gap E_g of the fractional quantized Hall state at $f = p/(2p+1)$, where p is an integer, to define the mass at energy scale E_g, through the asymptotic relation (presumed valid at large p) [1]

$$E_g \sim \frac{B\, e\, \hbar}{(2p+1)\, c\, m^*(E_g)}\,. \tag{1}$$

It is difficult to apply this expression to actual experimental data for the energy gap, however, because of the large effects of impurity scattering [12], for which there is no proper theory. Values of the effective mass have also been obtained recently from the amplitude of Shubnikov-de Haas oscillations at higher temperatures; however, these results are obtained using a theory of Shubnikov-de Haas oscillations based on ordinary Fermi liquid theory which may or may not be correct near $f = 1/2$ for the temperatures in question [13].

Within the composite fermion picture, at the mean field level, the ground state of the fractional quantized Hall state at $f = p/(2p+1)$ is just an integer quantized Hall state, with $|p|$ filled Landau levels, as originally noted by Jain. The effective field AB is equal to $B/(2p+1)$ in this case, and the gap (1) is identified with the cyclotron energy of a particle of mass m^* in the field AB. In order to properly obtain the dispersion relation for neutral excitations (quasi-exciton modes) or to study the linear response functions at finite q and w, it is necessary to go beyond the mean field approximation, at least to the level of random phase approximation, or, better, to modifications of the RPA based on Landau Fermi-liquid theory [1,6,14].

A derivation of the fermion Chern-Simons picture is given in Appendix A below.

III. EFFECTS OF DISORDER

A different aspect of the unquantized Hall effect is the transition from one quantized Hall plateau to another under conditions where impurity scattering plays a dominant role. When impurities are important we must distinguish more carefully between the filling fraction f, which is defined in terms of the electron-density n_e by

$$f = \frac{n_e hc}{Be}\,, \tag{2}$$

and the dimensionless Hall conductance ν, which is defined by

$$\sigma_{xy} = \frac{\nu e^2}{h}\,. \tag{3}$$

If the impurity scattering is sufficiently strong, relative to the electron-electron interaction, there should be a ***direct transition*** from one integer Hall plateau to another, in an *interval of* magnetic *fields* (or of f) that becomes *infinitely narrow* in limit of temperature $T \rightarrow 0$. If the impurity scattering is reduced, then the first fractional Hall plateau appears, say at $\nu = 1/3$. In this case, according to recent theoretical analyses [15], we should find a reentrant phase diagram, where in the limit of $T \rightarrow 0$, *as* the magnetic field decreases and f increases, we find a sequence of sharp transitions: first from an insulating state with $\nu = 0$ to a fractional quantized Hall state with $\nu = 1/3$, then back down to $\nu = 0$, followed by a sharp transition to $\nu = 1$. (A direct transition from $\nu = 1/3$ to $\nu = 1$ is not allowed.) If the impurity scattering is decreased further, we should expect to see new plateaus appear at $\nu = 2/3, \nu = 2/5$, etc. If the impurity scattering is reduced sufficiently, we reach the situation where at least for the lowest attainable temperatures, there is a smooth variation of the Hall conductance near $f = 1/2$, and the phenomena characteristic of an impurit.y-free system begin to appear.

Recent theoretical work has dealt with such questions as: Which transitions are allowed to be direct at $T = 0$? What are the transition widths at $T \neq 0$? What is the value of ρ_{xx} at the peak of the transition [15-17]? Other work has focused on the conductivity at $T \neq 0$ in the field region of a quantized Hall plateau: and on the possibility that mesoscopic density inhomogeneities may be responsible for the observed values of ρ_{xx} at higher temperatures, where Hall plateaus have disappeared [18,19].

IV. EDGE STATES

If a sample is in a quantized Hall state, so that there is an energy gap for delocalized excitations in the "bulk" of the sample, there must nevertheless be zero-energy excitations (edge states) along the sample boundaries. For noninteracting electrons, in an integer quantized Hall state, the edge states may be understood as arising because the occupied Landau levels are pushed up through the Fermi level by the confining potential at the boundary. (See Fig. 1, left panel). Electrons in edge states at a given boundary have a group velocity, parallel to the boundary, in a single direction. arising from $\vec{E} \times \vec{B}$ drift of the orbits in the electric field of the confining potential [20]. An alternate view is to think about each edge state as the dividing line between two incompressible quantum Hall fluids with different values of ν. An extra particle at the edge causes a bulge in the boundary, which propagates with a well-defined velocity and direction as an "edge magnetoplasmon" [21-23]. (See right panel of Fig. 1).

Recent theories have extended these considerations to fractional quantized Hall states, and have characterized the possible combinations of edge states that may occur [24-27]. In the fractional case, quantum fluctuations have a major effect, particularly on the tunneling of charged quasiparticles into or out of an edge state. The effects of these fluctuations can be understood using the various techniques previously applied to conventional one-

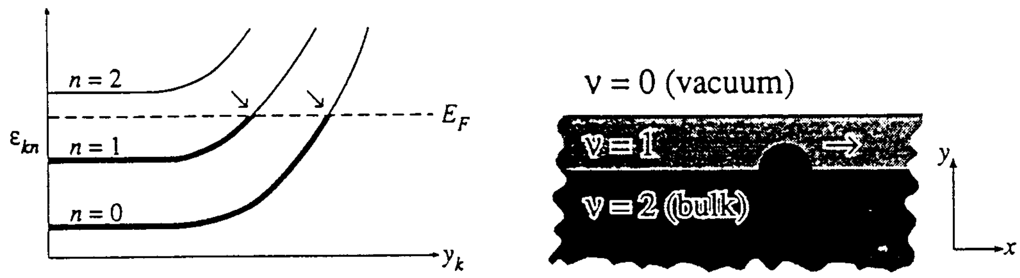

FIG. 1. Alternate descriptions of edge states, for noninteracting spinless electrons, with two filled Landau levels. The left panel shows the single-electron energy spectrum, ϵ_{kn}, in the Landau gauge, near a sample boundary. The wave function with wavevector k in the x direction is localized about $y = y_k \equiv k\hbar c/Be$. Heavy lines indicate occupied states; arrows point to edge states at the Fermi level. The right panel shows regions of space where there are respectively 2, 1 and 0 occupied Landau levels. An electron added to the inner edge state is here indicated as a bulge, which propagates to the right, as indicated. Electron-electron interactions modify the velocities, but not the directions of propagation.

dimensional metals, and the resulting description of the edge is characterized as a "chiral Luttinger liquid" [22,27]. Recent tunneling experiments confirm key predictions of the theory [28].

In most actual samples, the electron profile drops gradually to zero at the edge, on a scale large compared to the electron-electron separation. In a number of recent papers, the authors have attempted to calculate self-consistently the electron profile near a sample edge, and have discussed some of the differences that may occur between properties of gradual and sharp edges [23,29].

V. BILAYER SYSTEMS

A variety of interesting theoretical and experimental questions arise when there are two parallel electron layers. (A wide single quantum well may also act like a two layer system because the self-consistent Coulomb potential develops a peak at the center of the well.) Depending on the separation between the layers, the Coulomb interaction between electrons in different layers may be relatively weak or may become comparable in strength to the interaction between electrons in the same layer. Depending on the height as well as the thickness of the barrier between the layers, tunneling of electrons between the two layers may be more or less important. Depending on these parameters, and on the filling factors of the layers, a variety of quantized Hall states, as well as unquantized states, have been found to occur [30-33].

One interesting experimental result has been the observation of a quantized Hall plateau at total filling $f = 1/2$ for certain ranges of the system parameters [30-32]. Another interesting phase can occur at total fillings $f = 1$, (and at various other filling factors), where the total filling is locked at a quantized value, but the relative fraction of electrons in each layer is free to vary. Interest has focussed on characterizing the possible phases that can occur, and understanding their properties, as well as on predicting the occurrence of transitions between different phases as the system parameters are varied at a fixed filling factor f [30-36].

Application of a magnetic field parallel to the surface introduces, effectively, a spatial variation in the phase of the tunneling matrix element between the two layers. An unexpected phase transition, observed in a bilayer system with $f = 1$, in a relatively weak parallel field, has been explained in terms of this effect [33,37].

VI. OTHER TOPICS

Theoretical progress has also been made on a variety of other aspects of two-dimensional electron systems in strong magnetic fields. I cite but a few examples.

In any given sample, for *sufficiently strong magnetic fields,* at low enough temperatures, one expects to find an *insulating* phase where ρ_{xx} becomes very large, and ρ_{xy} is small compared to ρ_{xx}. This could occur because of electron-electron interaction (formation of a Wigner crystal), because of electron impurity interactions (carrier freeze-out) or because of some complicated combination of these effects. There has been considerable theoretic&l and experimental effort concerning the transition to the insulating state under various circumstances [38].

Theoretical arguments have been advanced to explain the observations of a pseudogap for tunneling into a layer (or between two layers) in the unquantized Hall regime [39].

Electron-electron interactions have been invoked to explain the splittings and shifts of the cyclotron resonance line at very low filling factors [40].

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APPENDIX A. DERIVATION OF THE FERMION CHERN-SIMONS THEORY

The fermion Chern-Simons theory is based on a unitary transformation (a "singular gauge transformation") which was introduced by Leinaas and Myrheim in 1977 in their discussion of the possibility of fractional statistics in two-dimensional systems [41]. Here, we consider a twodimensional system of spinless electrons, governed by a Hamiltonian of the form

$$H = K + V \, , \tag{A.1}$$

where K is the kinetic-energy operator,

$$K \equiv \frac{1}{2m_b} \int d^2r \psi_e^+(r)[-i\nabla + eA(r)]^2 \, \psi_e(r) \, , \tag{A.2}$$

and V is a potential-energy operator which depends only on the positions of the electrons. The operator $\psi_e^+(r)$ is the creation operator for an electron at point r, the vector potential $A(r)$ is due to a uniform external magnetic field B which points in the z direction normal to the plane, and m_b is the band mass of the electrons. We employ units where $\hbar$ = c = 1, and the electron charge is -e.

We next introduce a "quasiparticle" creation operator $\psi^+(r)$ which is related to ψ_e^+ by

$$\psi^+(r) = \psi_e^+(r) \exp \left[-i\tilde{\phi} \int d^2r' \arg(r - r')\rho(r') \right] \, , \tag{A.3}$$

where $\arg(r - r')$ is the angle the vector $(r - r')$ forms with the x axis, and

$$\rho(r) \equiv \psi_e^+(r)\psi_e(r) = \psi^+(r)\psi(r) \tag{A.4}$$

is the density of particles at point r. The operators $\psi^+(r)$ and $\psi(r)$ will obey the usual fermion commutation relations provided that $\tilde{\phi}$ is an even integer. In terms of the transformed operators $\psi^+(r)$, we may write the kinetic operator in the following form:

$$K = \frac{1}{2m_b} \int d^2r \psi^+(r)[-i\nabla + eA(r) - a(r)]^2 \psi(r) \, , \tag{A.5}$$

where

$$\mathrm{u(r)} \; 3 \; \tilde{\phi} \int d^2r' g(r - r')\rho(r') \tag{A.6}$$

$$\mathrm{g(r)} \; 3 \; (\hat{z} \times r)/r^2 . \tag{A.7}$$

For the potential energy *V*, *we* assume the usual two-body Coulomb interaction, which has the same form when expressed in terms of either ψ or ψ_e.

The vector potential defined by (A.6) and (A.7) is often referred to as a Chern-Simons field, because in a Lagrangian formulation, it is obtained by including a Chern-Simons term in the Lagrangian [6]. It follows from Eq. (A.6) that the Chern-Simons magnetic field operator,

b(r) ≡ v x a(r) , (A.8)

is related to the particle density operator by

b(r) = 27;&(r) . (A.9)

As a starting point, we define a mean-field Hamiltonian [42]

$$H_0 = \frac{1}{2m_b} \int \psi^+[-i\nabla + e\Delta A(r)]^2 \, \psi d^2 r \, , \quad \text{(A.1O)}$$

where AA is a mean-field vector potential which satisfies

V x (AA) = AB $\equiv B - 2\pi\bar{\phi} n_e/e$, (A.11)

where n_e is the mean value of the electron density $\rho(r)$.

At filling fraction $f = 1/2$, there are precisely two quanta of flux of the external magnetic field for each electron in the system. If we choose $\bar{\phi} = 2$, then there are two flux quanta of the fictitious "Chern-Simons" field V x $a(r)$ attached to each electron and the effective field AB = 0. Then the ground state of H_0 is just a filled Fermi sea [1], with Fermi wave vector

$$k_F = (4\pi n_e)^{1/2} \, . \quad \text{(A.12)}$$

Note, also, that if $f = p/(2p+1)$, where *p is a* positive or negative integer, then the effective field AB is given by

AB $= 2\pi n_e/pe$. (A.13)

Therefore the mean-field ground state is an integer quantized Hall state, with $|p|$ filled Landau levels in the effective field *AB*.

It should be understood that the mean-field ground state is only a first approximation to the true ground state. The central assumption of the fermion Chern-Simons theory is that for appropriate filling factors the correct properties of the system may be obtained from the mean-field theory by some type of perturbation analysis. The perturbation which must be used is the difference between the true (transformed) Hamiltonian and the mean field Hamiltonian H_0; *i.e.,* it is the sum of the two-body Coulomb interaction and the fluctuations of the Chern-Simons vector potential about its average value.

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