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Nontrivial Exactly Solvable Potentials with Linear Equations of Motion

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Based on an equations-of-motion method in the framework of Heisenberg's matrix mechanics, we investigate the conditions under which a one-dimensional quantum mechanical system becomes exactly solvable. By linearizing the equation of motion which is a double-commutation relation of some appropriately chosen function of the position operator with the Hamiltonian, we obtained a set of nontrivial exactly solvable potentials in one dimension. These potentials not only can be solved analytically in closed forms but also contain both the Morse potential and the Poschl-Teller potential as their limiting cases. They may thus be valuable for some potential model calculations as well as for testing various approximation schemes. We also examine these potentials in the framework of supersymmetric quantum mechanics, which is particularly useful for studying exactly solvable potentials.

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I. INTRODUCTION

Exactly solvable potentials are exceptions rather than rules in quantum theory. The simple harmonic oscillator is an exactly solvable potential which plays an important role in many fields of physics. The Morse potential and the Poschl-Teller potential provide two more well-known examples of exactly solvable potentials in one dimension [1]. There are, however, rather few exactly solvable potentials even in one dimension.

Exactly solvable potentials are useful because their energy eigenvalues as well as eigenfunctions can be evaluated analytically in closed forms. They can not only be used as potential models for some simplified physical systems but also be utilized as a convenient testing ground for various approximation methods. Therefore, it would be very desirable if one could find some more exactly solvable potentials besides those already known. The main purpose of the present paper is thus to develop a straightforward procedure to exploit the possibility of finding nontrivial exactly solvable potentials in one-dimensional quantum theory.

Based on what we have learned from the simple harmonic oscillators and using the equations-of-motion (EOM) method within the framework of Heisenberg's matrix mechanics [2,3], we force the double-commutator equation of motion for some appropriately chosen function of the position operator to be linear in that function. Among all the potentials in one dimension that allow such linear equations of motion, we are able to obtain a set of exactly solvable potentials which are nontrivial. These potentials have the nice feature that they not only can be solved analytically in closed forms but also contain both the Morse potential and the Pöschl-Teller potential as their limiting cases. They may therefore be valuable for some potential model calculations or for testing various approximation schemes in quantum theory.

Here we will first investigate the conditions under which an arbitrary function of the position operator will possess a linear double-commutation relation with the Hamiltonian, *i.e.,* a linear double-commutator equation of motion. Since a linear double-commutator equation of motion automatically gives rise to an exactly solvable potential, we will examine what type of exactly solvable potentials can actually be derived this way. We will then use the EOM method in the framework of Heisenberg's matrix mechanics to evaluate the energy eigenvalues together with some matrix elements of interest analytically for these potentials. Finally, we will also study these potentials in the framework of supersymmetric quantum mechanics (SQM) [4], which has become a rather useful tool in quantum theory in recent years, especially in dealing with exactly solvable potentials.

II. LINEAR DOUBLE-COMMUTATOR EQUATIONS OF MOTION

We consider the one-dimensional quantum system with the Hamiltonian

$$H = \frac{p^2}{2} + V(x), \tag{1}$$

where the mass has been set to 1, and by using the units that $\hbar = 1$, the position and momentum operators satisfy the commutation relation

$$[x,p] = i. \tag{2}$$

For this one-dimensional quantum system the double-commutator equation of motion (EOM) for x is simply the double-commutation relation

$$[[x,H],H] = V', \tag{3}$$

where $V' = dV(x)/dx$. In the case of the simple harmonic oscillator with $V(x) = \frac{1}{2}kx^2$, it is easy to see that the right hand side of Eq. (3) is simply kx and thus the double-commutator

$[[x, H], H]$ is linear in x. The fact that the simple harmonic oscillator is exactly solvable is a consequence of the linearity of this double-commutator EOM for the position operator. For other potentials, the right hand side of Eq. (3) is no longer linear in x. However, this does not mean that all potentials other than the simple harmonic oscillator can not be solved exactly in closed forms. In fact, the Morse potential and the Poschl-Teller potential are two other well-known exactly solvable potentials in one dimension besides the simple harmonic oscillator.

In Ref. [2], we examined some structural and numerical aspects of one-dimensional quantum systems in the framework of Heisenberg's matrix mechanics using the EOM method developed previously [3]. We have shown there that both the Morse and the Poschl-Teller potentials also possess a linear double-commutator EOM, but of course not for the position operator x. In the case of the Morse potential $V(x) = \lambda(e^{-2qx} - 2e^{-qx})$, the linear double-commutation relation is $[[e^{qx}, H], H] \sim e^{qx} + c$ while in the case of the Poschl-Teller potential $V(x) = -\lambda/\cosh^2 qx$, it is $[[\sinh qx, H], H] \sim \sinh qx + c$, where c is a constant. Here, we will like to see whether there are other exactly solvable potentials which also allow a linear double-commutator EOM for some appropriately chosen function of the position operator.

To do that, we first derive the following double-commutation relation with the Hamiltonian for an arbitrary function of the position operator, say f = f(z). Namely,

$$[[f, H], H] = -(f''H + Hf'') + 2f''V + f'V' - \frac{1}{4}f^{(4)}, \tag{4}$$

where f', f'' and $f^{(4)}$ denote respectively the first, second and fourth derivatives of f with respect to x. In order to make the double commutator $[[f, H], H]$ linear in f, we set

$$2f''V + f'V' = \alpha f + \beta\,, \tag{5}$$

and

$$f'' = \mu f + \nu\,, \tag{6}$$

where α, β, μ and ν are arbitrary constants. Eq. (6) is a condition to be satisfied by the function **f(z),** while Eq. (5) is a condition to be imposed on the potential $V(x)$. In fact, Eq. (5) can be easily solved to yield

$$V(x) = \frac{1}{2(f')^2}\{\alpha f^2 + 2\beta f + \gamma\} \tag{7}$$

with

$$(f')^2 = \mu f^2 + 2\nu f + \sigma\,, \tag{8}$$

which readily follows from Eq. (6). In Eqs. (7) and (8), γ and σ are integration constants obtained through integrating Eqs. (5) and (6), respectively. For $V(x)$ and $f(x)$ satisfy Eqs. (5) and (6), or equivalently Eqs. (7) and (8), the double-commutator EOM in Eq. (4) becomes

$$[[f,H],H] = \left(\alpha - \frac{\mu^2}{4}\right) f - \mu(fH + Hf) - 2\nu H + \beta - \frac{\mu\nu}{4}\,, \tag{9}$$

which is clearly linear in f because we have imposed the conditions given in Eqs. (5) and (6) respectively on the potential and the function $f(z)$.

In Sec. IV we will use the EOM method in the framework of Heisenberg's matrix mechanics [2,3] to evaluate analytically the energy eigenvalues as well as the matrix elements of $f(z)$ between exact eigenstates of the Hamiltonian. Utilizing Eq. (7), it is useful to rewrite the Hamiltonian as

$$H = \frac{p^2}{2} + \frac{1}{2(f')^2}\{\alpha f^2 + 2\beta f + \gamma\}\,. \tag{10}$$

To facilitate the evaluation of the energy eigenvalues, one multiplies both sides of Eq. (10) with $(f')^2$ to obtain

$$\begin{aligned}\{(f')^2, H\} &= [f,H][H,f] - \frac{1}{2}f'f''' - \frac{3}{4}(f'')^2 + 2(f')^2 V \\ &= [f,H][H,f] + \left(\alpha - \frac{5}{4}\mu^2\right)f^2 + 2\left(\beta - \frac{5}{4}\mu\nu\right)f \\ &\quad +\gamma - \frac{3}{4}\nu^2 - \frac{\mu\sigma}{2}\,,\end{aligned} \tag{11}$$

where $f''' = d^3f/dx^3$ and $\{\,,\}$ denotes an anti-commutator. Note that Eqs. (6)-(8) have been employed to establish the last equality above. Another useful relation is

$$[f,[H,f]] = (f')^2 = \mu f^2 + 2\nu f + \sigma\,, \tag{12}$$

which is indispensable for evaluating the energy eigenvalues as well as the matrix elements of $f(x)$.

Before we proceed to evaluate the energy eigenvalues, we will first examine what exactly solvable potentials are actually implied by Eqs. (6)-(8). For this we now turn to the next section.

III. EXACTLY SOLVABLE POTENTIALS GIVEN BY LINEAR DOUBLE-COMMUTATOR EOM

To see what exactly solvable potentials are actually implied by Eqs. (6)-(8), it is heuristic to list the following special cases:

$$\mathrm{j}(2) = x \longrightarrow \mathrm{V}(\mathrm{z}) = \frac{1}{2}(\alpha x^2 \ \mathrm{t}\ 2\beta x + \gamma), \tag{13a}$$

corresponding to a simple harmonic oscillator with the origin shifted from $x = 0$ to $x = -\beta/\alpha$;

$$f(x) = x^2 \longrightarrow V(x) = \frac{1}{8}\left(\alpha x^2 \ \mathrm{t}\ 2\beta + \frac{-}{a}\right), \tag{13b}$$

corresponding to a three-dimensional harmonic oscillator if one sets 7 = 4 $l(l$ • l- 1) $\hbar^2/m$;

$$\mathrm{f}(\mathrm{x}) = \mathrm{e}^{qx} \longrightarrow V(x) = \lambda(\mathrm{e}^{-2qx} - 2\,\mathrm{e}^{-qx}), \tag{13c}$$

which is the Morse potential, if one sets $\alpha = 0$, $\beta = -4q^2\lambda$ and $\gamma = 2q^2\lambda$; and,

$$\mathrm{j}(\mathrm{x}) = \mathrm{sinhqx} \longrightarrow V(x) = -\frac{\lambda}{\cosh^2 qx}, \tag{13d}$$

which is the Poschl-Teller potential, if one sets $\alpha = \beta = 0$ and $\gamma/2q^2 = -\lambda$.

What we have given above are simply the well-known examples of exactly solvable potentials in one dimension. To obtain more general potentials, one has to examine more closely the condition imposed on the function f(x) given in Eq. (6) or equivalently in Eq. (8). In fact, there are two types of solutions to Eq. (6) or (8) corresponding respectively to $\mu = 0$ and $\mu \neq 0$. Namely,

$$f = ax^2 \cdot \mathrm{t}\ bx + \mathrm{c}, \qquad (\mu = 0), \tag{14a}$$

and

$$f = A\,\mathrm{e}^{\sqrt{\mu}x} + B\,\mathrm{e}^{-\sqrt{\mu}x} + C, \qquad (\mu \neq 0), \tag{14b}$$

in which a, *b* and c as well as *A*, *B* and C are arbitrary constants. It is not difficult to see that Eq. (14a) yields only the harmonic oscillators, as given in Eqs. (13a) and (13b), so it is not of interest here. On the other hand, the solution with $\mu \neq 0$ given in Eq. (14b) turns out to quite interesting. Through Eq. (7), it gives rise to the potential

$$V(x) - \frac{\beta(A\,\mathrm{e}^{\sqrt{\mu}x} + B\,\mathrm{e}^{-\sqrt{\mu}x}) + \gamma/2}{\mu(A\,\mathrm{e}^{\sqrt{\mu}x} - B\,\mathrm{e}^{-\sqrt{\mu}x})^2}. \tag{15}$$

Note that in obtaining Eq. (15) the constants a in Eq. (7) and C in Eq. (14b) have been set to zero. This does not cause any loss of generality in the present case of using the solution of f with $\mu \neq 0$ because, for instance, αf^2 in Eq. (7) contributes only an insignificant constant term to the potential together with a term which can always be taken care of by redefining the parameter 7. Comparing Eqs. (14b) and (15) with Eqs. (13c) and (13d), it can be easily seen that Eq. (15) reduces to the Morse potential if one sets $B = 0$ and $\gamma/\mu = -\beta A/\mu =$ 2X and to the Poschl-Teller potential if one sets $A = -B = 1/2$,

$\beta = 0$, and $\gamma/2\mu = -\lambda$. The potential given above in Eq. (15) thus contains both the Morse potential and the Poschl-Teller potential as its limiting cases. In other words, what we have obtained here by forcing the double commutator $[[f(x), H], H]$ to be linear in $f(x)$ is a family of exactly solvable potentials more general than both the Morse potential and the Poschl-Teller potential. Eq. (15) constitutes the main result of the present endeavor of finding nontrivial exactly solvable potentials, using the linear equations-of-motion method.

The potential as given by Eq. (15) vanishes as $|x|$ goes to ∞ if $\mu > 0$. The complete set of states always includes the continuum as part of the spectrum. Here we have of course assumed that the parameters A, $B, \beta/\mu$ and γ/μ are chosen such that the potential allows some bound states of negative energies. Since in the present case the continuum part of the spectrum consists of all energies from 0 to ∞, only the bound-state energy eigenvalues are of interest here. If, on the other hand, $\mu < 0$, the spectrum may consist of bound states only, as can be seen in the case of the Pöschl-Teller-type potentials, which are periodic infinite potential wells because a cosine function instead of a hyperbolic cosine function is now in the denominator. In the next section, we will evaluate analytically the bound-state energy eigenvalues of these exactly solvable potentials. For convenience, we will use the form of the potentials as originally given in Eqs. (7) and (8) and assume that the parameters therein are appropriately chosen so that there are always some bound states allowed by these potentials.

IV. EXACT BOUND-STATE ENERGY EIGENVALUES AND MATRIX ELEMENTS OF f(z)

In what follows we will use the equations-of-motion method [2,3] to evaluate analytically the energy eigenvalues for bound states as well as the matrix elements of $f(x)$ in the framework of Heisenberg's matrix mechanics. Consider the Hamiltonian H of Eq. (10) with the potential explicitly given in Eqs. (7) and (8). We shall work in the basis formed by the eigenstates of the Hamiltonian. Namely, with E_n and $|n\rangle$ denoting the exact energy eigenvalues and eigenstates, respectively, one has

$$H|n\rangle = E_n|n\rangle . \tag{16}$$

First, we take the matrix elements of Eq. (9) between $\langle n|$ and $|n'\rangle$ to obtain

$$\left\{(E_{n'} - E_n)^2 + \mu(E_n + E_{n'}) - \alpha + \frac{\mu^2}{4}\right\} f_{nn'} = \delta_{nn'}\left\{\beta - \frac{\mu\nu}{4} - 2\nu E_n\right\}, \tag{17}$$

where the notation $f_{nn'} = \langle n|f|n'\rangle$ has been used. The important thing to note here is that Eq. (17) implies that the only non-vanishing matrix elements of $f(x)$ are diagonal matris elements ($n' = n$) and those between neighboring states ($n' = n \pm 1$). Thus, for $n' = n + 1$ Eq. (17) yields

$$(E_{n+1}-E_n)^2+\mu(E_{n+1}+E_n)=\alpha-\frac{1}{4}\mu^2\,, \tag{18}$$

while for $n'=n$ it gives rise to

$$f_{nn}=\frac{\beta-\frac{1}{4}\mu\nu-2\nu E_n}{2\mu E_n-\alpha+\frac{1}{4}\mu^2}\,. \tag{19}$$

The diagonal matrix elements of Eq. (12) yield

$$[2(E_{n+1}-E_n)-\mu]f_{n,n+1}^2=[2(E_n-E_{n-1})+\mu]f_{n-1,n}^2+\mu f_{n,n}^2+2\nu f_{nn}+\sigma\,. \tag{20}$$

To evaluate the energy eigenvalues, one needs only the diagonal matrix elements of Eq. (11) given by

$$\begin{aligned}&2E_n\{\mu(f^2)_{nn}+2\nu f_{nn}+\sigma\}\\&\quad=(E_{n+1}-E_n)^2f_{n,n+1}^2+(E_n-E_{n-1})^2f_{n-1,n}^2+\left(\alpha-\frac{5}{4}\mu^2\right)(f^2)_{nn}\\&\quad+2\left(\beta-\frac{5}{4}\mu\nu\right)f_{nn}+\gamma-\frac{3}{4}\nu^2-\frac{1}{2}\mu\sigma\,,\end{aligned} \tag{21}$$

where $(f^2)_{nn}=\langle n|f^2|n\rangle=f_{n,n+1}^2+f_{n-1,n}^2+f_{n,n}^2$.

Note that in the derivations above we have used extensively the fact that the only nonvanishing matrix elements of $f(x)$ are those $f_{nn'}$ with $n'=n$ and $n'\pm1$. And, from Eqs. (18)-(21), it is not difficult to obtain the following exact energy eigenvalues for bound states. Namely,

$$E_n=-\frac{1}{2\mu}\left\{\Gamma(\alpha,\beta,\gamma,\mu,\nu,\sigma)-\left(n+\frac{1}{2}\right)\mu\right\}^2+\frac{\alpha}{2\mu}\,, \tag{22a}$$

where $n=0,1,2,\cdots$ is bounded by $\Gamma(\alpha,\beta,\gamma,\mu,\nu,\sigma)/\mu-1/2$ if $\mu>0$, and

$$\begin{aligned}&\Gamma(\alpha,\beta,\gamma,\mu,\nu,\sigma)\\&\quad=\left\{\frac{1}{2}\left(\alpha+\frac{\mu^2}{4}-\frac{\mu\gamma'}{\sigma'}\right)\pm\frac{1}{2}\sqrt{\left(\alpha+\frac{\mu^2}{4}-\frac{\mu\gamma'}{\sigma'}\right)^2+\frac{4\mu\beta'^2}{\sigma'}}\right\}^{1/2},\end{aligned} \tag{22b}$$

with β', γ' and σ' given by

$$\beta'=\beta-\frac{\alpha\nu}{\mu}\,, \tag{23a}$$

$$\gamma'=\gamma-\frac{2\beta\nu}{\mu}+\frac{\alpha\nu^2}{\mu^2}\,, \tag{23b}$$

$$\sigma' = \sigma - \frac{\nu^2}{\mu} \,. \tag{23c}$$

In Eq. (22b), only the the positive sign is allowed if $\mu/\sigma' > 0$. When $\mu/\sigma' < 0$, the negative sign should be taken only if the positive sign would give rise to an E_0 lower than the minimum of the potential or would cause the ground state eigenfunction as given in Eq. (28) below to be unnormalizable.

V. SOLUTION BY SUPERSYMMETRIC QUANTUM MECHANICS

In this section we shall examine the nontrivial exactly solvable potentials obtained above in the framework of supersymmetric quantum mechanics (SQM) [4]. First, we rewrite the Hamiltonian given in Eq. (10) as

$$H = A^{\dagger} A + E_0 \,, \tag{24a}$$

where

$$A + = - \& \left\{ -\frac{d}{dx} + g(x) \right\} = (A)+. \tag{24b}$$

It is not difficult to see that Eq. (24) gives exactly the Hamiltonian of Eq. (10) if we use the ansatz

$$g(x) \quad {-}{=}\frac{af(x)+b}{f'(x)} \,, \tag{25a}$$

with a and b given by

$$a = -\frac{\mu}{2} + \Gamma \,, \tag{25b}$$

$$b = \frac{\beta - \nu\alpha/\mu}{\Gamma} + \frac{a\nu}{\mu} \,, \tag{25c}$$

where $\Gamma \equiv \Gamma(\alpha, \beta, \gamma, \mu, \nu, \sigma)$ has been defined in Eq. (22b). Furthermore, E_0 in Eq. (24) is just the ground state energy eigenvalue and is readily found to be

$$E_0 = -\frac{1}{2\mu}\left(\Gamma - \frac{\mu}{2}\right)^2 + \frac{\alpha}{2\mu} \,. \tag{26}$$

Note that the function f = f(z) in Eq. (25a) satisfies the conditions given in Eqs. (6) and (8) and we have assumed that $\mu > 0$. A nice feature of the SQM is that the ground state wavefunction $\psi_0(x)$ can be easily found by the condition

$$A\psi_0(x) = 0 \,, \tag{27}$$

which, with Co being an integration constant, gives rise to

$$\begin{aligned}\psi_0(x) &= C_0 \exp\left(-\int g\, dx\right)\\ &= C_0 |f'(x)|^{(1/2-\Gamma/\mu)}\\ &\quad \cdot \exp\left[-\frac{(\beta-\alpha\nu/mu)}{(\mu\sigma-\nu^2)\Gamma}\tan^{-1}\left(\sqrt{\mu^2/(\mu\sigma-\nu^2)}\ f(x)\right)\right].\end{aligned} \tag{28}$$

The SQM partner of the Hamiltonian given in Eq. (24) is simply

$$\widetilde{H} = \boldsymbol{AA}+ + E_0 = H + \frac{d}{dx}g(x), \tag{29}$$

which can be shown to be also in the form of the exactly solvable potentials given in Eq. (10). Thus, the above procedure as given by Eqs. (24)-(28) can be applied afresh to $\widetilde{H}$. Note that the first excited state eigenvalue E_1 and wavefunction ψ_1 of H are related to the ground state eigenvalue $\widetilde{E}_0$ and wavefunction $\widetilde{\psi}_0$ of $\widetilde{H}$ by $E_1 = \widetilde{E}_0$ and $\psi_1 \sim A^\dagger \widetilde{\psi}_0$, respectively. Other energy eigenvalues and eigenfunctions of H can be found in the same fashion by using similar relations between the energy eigenvalues and eigenfunctions of the various SQM partners of the Hamiltonian. The energy eigenvalues obtained this way are, of course, the same as what we have found by using the equations-of-motion (EOM) method in the frame work of Heisenberg's matrix mechanics. While the EOM method is useful in giving the various matrix elements of interest, the SQM is particularly useful in getting the energy eigenfunctions.

Before we conclude this section, it is worthwhile to point out that the nontrivial example of a quantum superpotential in the framework of SQM given in Ref. [5] is only a limiting case of the present exactly solvable potentials with f = sinh x and appropriate parameters a and b or α and β, etc.

VI. SUMMARY AND DISCUSSION

In this paper we have presented a linear equations-of-motion method for finding exactly solvable potentials. In this method one linearizes the double-commutation relation of some appropriately chosen function of the position operator with the Hamiltonian. We have found in this way a set of nontrivial exactly solvable potentials in one-dimensional quantum theory which contain both the Morse potential and the Pöschl-Teller potential as limiting cases. These potentials are interesting in their own right because their energy eigenvalues and eigenfunctions can be evaluated analytically in closed forms. Since these potentials are more general than the Morse potential and the Pöschl-Teller potential, we expect that they should be more useful also.

We have first evaluated the energy eigenvalues together with other physical qunatities for these exactly solvable potentials analytically using the equations-of-motion method in the framework of Heisenberg's matrix mechanics. A salient feature of these exactly solvable potentials is that they all possess a linear double-commutator equation of motion for some function of the position operator and this function has nonvanishing matrix elements only between the same or nearest-neighbor energy eigenstates. Furthermore, we have also studied these potentials using the techniques of supersymmetric quantum mechanics. While the supersymmetric quantum mechanics is powerful in getting the energy eigenvalues and eigenfunctions of exactly solvable potentials, the equations-of-motion method used here seems to be more handy in finding these potentials per se. Further investigation and application of this method, therefore, are warranted.

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