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Operator Algebra and Braid Group Structure in Chern-Simons Theory on a Torus†

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I. INTRODUCTION

In recent years there has been an intense interest in the theory of anyons, particles obeying fractional statistics in **1** + 2 dimensions [1]. This is so not only because the theory itself exhibits very interesting mathematical structures, but also because it could be of relevance to some condensed matter systems, notably the fractional quantum Hall effect and the high T_c superconductivity.

One way to implement fractional statistics in field theory is by introducing a Chern-Simons interaction among matter field, as is commonly done in models of anyon superconductivity. It has been demonstrated that when the coefficient of the Chern-Simons term is quantized in such a way that the anyon has statistics phase parameter $\theta_s = \pi/N$ with an integer N, the system of an anyon gas becomes superconducting. At present, the most probable candidate for high T_c superconductor requires $N = \pm 2$, which correspond to half-fermions, or semions.

There is no theoretical reason as to why anyon should assume the special values of statistics $\theta_s = \pi/N$ on a flat two-dimensional space. It is an interesting coincidence,

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however, that such values of the θ_s arise very naturally when the space is a Riemann surface of genus g ≥ 1 [2,3]. The requirement of manifest invariance under large gauge transformations leads to quantization of the coefficient of the Chern-Simons term at exactly the values that give rise to $\theta_s = \pi/N$. It turned out, however, that the consistency of a Chern-Simons theory coupled to matter fields does not necessarily demand the manifest gauge invariance, but requires a weaker condition that π/θ_s be a rational number [3,4] (For works related to Chern-Simons theory on multiply-connected spaces, see eg.: [5]).

On the other hand, the analysis (G-S] based on representation theory of braid group, which is essential in formulating quantum mechanics of anyons, has revealed that the global topology of the Riemann surface gives a constraint on the possible values of θ_s, the number of anyons, and the number of components of the wave functions. For a Riemann surface of genus g $\geq$ **1,** it is shown that fractional statistics is consistent only with multi-component wave functions.

The purpose of this talk is to demonstrate how the general results from braid group representation theory manifest themselves in a Chern-Simons field theory on a torus [9] and to present the algebraic relations among various operators, especially the Hamiltonian and the total momenta [10]. Special attention is given to the dynamics of the non-integrable phases of the two Wilson line integrals.

II. BOUNDARY CONDITIONS, TRANSLATION INVARIANCE, AND FLUX QUANTIZATION

We consider a Chern-Simons theory with non-relativistic electron field given by the Lagrangian

$$\begin{aligned} \mathcal{L} &= \frac{\kappa}{4\pi}\varepsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho + \psi^\dagger iD_0\psi - \frac{1}{2m_e}|D_k\psi|^2 , \\ D_0 &= \partial_0 + ia_0 , \qquad D_k = \partial_k - ia^k \end{aligned} \tag{2.1}$$

on a torus T^2 with fundamental domain $\mathcal{D}$ defined by $0 \leq x_j \leq L_j$, $j = 1,2$. The Chern-Simons coefficient κ is related to statistics parameter 8, by θ_s = n/n. The electron field $\psi(x)$ is taken to be fermionic for definiteness.

Equations of motion derived from (2.1) are

$$\frac{\kappa}{4\pi}\varepsilon^{\mu\nu\rho}f_{\nu\rho} = J^\mu , \tag{2.2}$$

where

$$\begin{aligned} j^0 &= \psi^\dagger\psi , \\ j^k &= -\frac{i}{2m_e}(\psi^\dagger D_k\psi - (D_k\psi)^\dagger\psi) , \end{aligned} \tag{2.3}$$

and

$$i\partial_0\psi = \left\{-\frac{1}{2m_e}D_k^2 + a_0\right\}\psi\,. \tag{2.4}$$

It follows from (2.2) that

$$-\frac{\kappa}{2\pi}\,b = n_e\,, \tag{2.5}$$

where $b = \partial_1 a^2 -$ &a ' $= -f_{12}$ etc.

Since the space is multiply-connected, one has to specify the boundary conditions of the fields. Translations along the two non-contractible loops are denoted by

$$\begin{aligned} &T_1 : \ (t,x,y) \rightarrow T_1(x) = (t, x + L_1, y)\,, \\ &T_2 : \ (t,x,y) \rightarrow T_2(x) = (t, x, y + L_2)\,. \end{aligned} \tag{2.6}$$

The most general boundary conditions are given by [II]

$$\begin{aligned} &a_\mu[T_j(x)] = a_\mu[x] + \partial_\mu\beta_j(x)\,, \\ &\psi[T_j(x)] = e^{-i\beta_j(x)}\,\psi(x)\,. \end{aligned} \tag{2.7}$$

In other words, the fields return to their original values up to gauge transformations after translations along the non-integrable loops. The two operators T_1 and T_2 need only to commute with each other up to a gauge transformation:

$$\begin{aligned} &a_\mu[T_1\cdot T_2(x)] = a_\mu[T_2\cdot T_1(x)] + \partial_\mu\gamma(x)\,, \\ &\psi[T_1\cdot T_2(x)] = e^{-i\gamma(x)}\,\psi[T_2\cdot T_1(x)]\,. \end{aligned} \tag{2.8}$$

This requirement leads to the following conditions on the β's and 7:

$$\{\beta_1(T_2x) - \beta_1(x)\} - \{\beta_2(T_1x) - \beta_2(x)\} = -\gamma(x)\,. \tag{2.9}$$

Further constraints on $\beta_j(x)$'s follow from the gauge invariance and smoothness of the fields. First consider the flux on a torus. (2.7) and (2.9) lead to $\Phi = \gamma(\boldsymbol{r}_0)$, where $\boldsymbol{r}_0 = (x_0, y_0)$. Since f_{12} is single-valued on the torus, Φ must be To-independent. This implies that $\gamma(\boldsymbol{r}_0)$ must be constant, $\gamma(x)$ = 7. A stronger condition follows from the smoothness of the field operators. Given the Lagrangian (2.1), the action principle, or the field equations, dictates that all fields are smooth functions of the coordinate $x = (t, \boldsymbol{r})$ of the covering space. On the covering space the two paths T_1T_2 and T_2T_1 *are* homotopically equivalent so that the continuity of physics implies that

$$\psi(T_1T_2x) = \psi(T_2T_1x)\,. \tag{2.10}$$

In other words, although the field operator $\psi(x)$ is multi-valued on a torus, it is single-valued on its covering space. We thus have

$$\begin{aligned} &\gamma = 0 \qquad (\text{mod } 27\text{r}) \\ &\Phi = 2\pi m \ (\text{m : integer}) \ . \end{aligned} \tag{2.11}$$

We see that the flux Φ is quantized on the torus. Typical β_j's which solve (2.9) with the flux (2.11) are

$$\begin{aligned} \beta_1(\boldsymbol{r}) &= -\frac{\pi m y}{L_2} \ , \\ \beta_2(\boldsymbol{r}) &= +\frac{\pi m x}{L_1} \ . \end{aligned} \tag{2.12}$$

These boundary conditions will be taken in the rest of the talk. Residual gauge transformations which maintain the boundary conditions are given by

$$\begin{aligned} &\Lambda(x) = \Lambda^{\text{large}}(x) + \tilde{\Lambda}(x) \ , \\ &\Lambda^{\text{large}}(x) = -\frac{2\pi n_1 x}{L_1} - \frac{2\pi n_2 y}{L_2} \qquad (n_1, n_2 : \text{ integers}) \ , \\ &\tilde{\Lambda}(T_j x) = \tilde{\Lambda}(x) \ . \end{aligned} \tag{2.13}$$

$\Lambda^{\text{large}}(x)$ is called a large gauge transformation.

One of the Chern-Simons Eq. (2.2) imposes another constraint:

$$q = \int dx \psi^\dagger \psi = -\int dx \frac{\kappa}{2\pi} b = -\frac{\kappa}{2\pi}\Phi \ .$$

Here q is the total number of particles residing on the torus. Hence

$$\frac{1}{2\pi}\Phi = -\frac{q}{\kappa} \ . \tag{2.14}$$

In the quantized theory, this equation becomes a constraint on the physical state:

$$\left(Q + \frac{\kappa}{2\pi}\Phi\right)|\text{phys}\rangle = 0 \ , \tag{2.15}$$

where $Q = \int dx j^0$. In any case, this condition implies that the Chern-Simons coefficient must have fractional values, $\kappa = -q/m$, in the presence of matter. Also, with the identification $\theta_s = \pi/\kappa$, this condition reproduces the constraint obtained by Einarsson from the investigation of the braid group on a torus:

$$\theta_s = -\frac{m\pi}{q} \ . \tag{2.16}$$

Let us take the boundary conditions (2.7) and (2.12). The Chern-Simons Eq. (2.2) can be solved in a gauge $\nabla \cdot a = 0$. The solution is [2]

$$
\begin{aligned}
a^j(x) &= \frac{\theta_j(t)}{L_j} - \frac{\Phi}{2L_1L_2}\epsilon^{jk}x_k + \hat{a}^j(x)\,, \\
\hat{a}^j(x) &= \frac{2\pi}{\kappa}\int dy\epsilon^{jk}\partial_k^x G(x-y)\Big(j^0(y) + \frac{\kappa\Phi}{2\pi L_1L_2}\Big)\,, \\
a_0(x) &= -\frac{2\pi}{\kappa}\int dy G(x-y)(\partial_1 j^2 - \partial_2 j^1)(y)\,,
\end{aligned}
\tag{2.17}
$$

where $G(\boldsymbol{r})$ is the periodic Green's function on a torus:

$$
\begin{aligned}
&\nabla^2 G(\boldsymbol{r}) = \delta(\boldsymbol{r}) - \frac{1}{L_1L_2}\,, \\
&G(T_j\boldsymbol{r}) = G(\boldsymbol{r})\,.
\end{aligned}
\tag{2.18}
$$

In the expression of $a_0(x)$ in (2.17), the current j^k contains $a^k(x)$ which is expressed in terms of ψ and $\psi^\dagger$.

The zero-modes of $a^j(x)$, θ_j, are the non-integrable phases of the Wilson line integrals on the torus. They cannot be eliminated by the residual gauge transformations (2.13).

III. OPERATOR ALGEBRA

In the quantum theory, physical degrees of freedom of the system are the matter fields $\psi(x)$ (here taken to be fermionic) and the non-integrable phases θ_j's. The matter fields obey

$$
\begin{aligned}
&\{\psi(t,\boldsymbol{r}), \psi(t,\boldsymbol{r}')\} = 0\,, \\
&\{\psi(t,\boldsymbol{r}), \psi^\dagger(t,\boldsymbol{r}')\} = \delta(\boldsymbol{r}-\boldsymbol{r}')\,, \qquad \text{for } \boldsymbol{r},\boldsymbol{r}' \in \mathcal{D}\,.
\end{aligned}
\tag{2.19}
$$

Insertion of (2.17) into the Lagrangian (2.1) gives

$$
\int_{\mathcal{D}} d\boldsymbol{r}\mathcal{L} \Longrightarrow \frac{\kappa}{4\pi}(\theta_2\dot{\theta}_1 - \theta_1\dot{\theta}_2)\,.
$$

In other words, θ_1 and θ_2 become canonically conjugate to each other [2], satisfying

$$
[\theta_1, \theta_2] = \frac{2\pi i}{\kappa}\,. \tag{2.20}
$$

The Hamiltonian and total momentum operators are given by

$$
\begin{aligned}
H &= \frac{1}{2m}\int dx (D_k\psi)^\dagger(D_k\psi)\,, \\
P^k &= -i\int dx\psi^\dagger D_k\psi\,.
\end{aligned}
\tag{2.21}
$$

It should be noted that there is an ordering ambiguity in writing the Hamiltonian (2.21) in terms of the dynamical variables with the aid of (2.17). We shall take the ordering of ψ and

$\psi^\dagger$ as it is in (2.21) The equations of motion for ψ and θ_j can then be worked out [10]. It is also found that the relation (2.14) does not follow from the Hamiltonian and commutation relations. As mentioned before, it has to be imposed as a constraint on the physical states.

There are two other sets of important operators in the theory. These are the Wilson line operators, $W_j = e^{i\theta_j}$, and the two generators of large gauge transformations:

$$U_j = \exp\left\{i\epsilon^{jk}\kappa\theta_k - 2\pi i \int dx \frac{x_j}{L_j} \psi^\dagger\psi(x)\right\}. \tag{2.22}$$

W_j and U_j satisfy dual relations:

$$W_1 W_2 = e^{-2\pi i/\kappa} W_2 W_1 , \tag{2.23}$$

and

$$U_1 U_2 = e^{-2\pi i\kappa} U_2 U_1 . \tag{2.24}$$

The Wilson line integrals W_j, as well as H and P^k, are gauge invariant so that

$$[U_j, W_k] = [U_j, P^k] = [U_j, H] = 0 . \tag{2.25}$$

One can compute the commutation relations among H, P^k and W_j:

$$\begin{aligned}
[P^j, P^k] &= i\epsilon^{jk}\frac{2\pi}{\kappa L_1 L_2} Q\left(Q + \frac{\kappa}{2\pi}\Phi\right), \\
[P^j, H] &= i\epsilon^{jk}\frac{2\pi}{\kappa L_1 L_2} J^k\left(Q + \frac{\kappa}{2\pi}\Phi\right), \\
[W_j, P^k] &= \epsilon^{jk}\frac{2\pi}{\kappa L_k} Q W_j , \\
[W_j, H] &= \epsilon^{jk}\frac{\pi}{\kappa L_k}(J^k W_j + W_j J^k),
\end{aligned} \tag{2.26}$$

where $J^k \equiv \int dx j^k = P^k/m$. From these equations one sees that W_j's map an eigenstate into another eigenstate corresponding to different momenta and energy. Also, it should be emphasized that H and P^k do not commute with each other as operators, but only commute in the physical Hilbert space.

It is interesting to note here that the operator algebra (2.26) remain intact in Chern-Simons gauge theory coupled to Dirac matter field, though in this case J^k is not conserved even in the physical Hilbert space. Therefore W_j no longer maps an eigenstate of H into another.

IV. VACUUM

In the Chern-Simons theory on a torus the vacuum has a non-trivial structure. Physical states, particularly the vacuum, must transform among themselves under the operators U_j and W_j. It is known that consistent theory can be constructed provided that (U_1, U_2) and (W_1, W_2) satisfy the Weyl-Heisenberg algebra (2.23) and (2.24) with $\kappa = p/q$ where p and q are coprime integers [3,4].

From now on we shall concentrate on the case $q = 1, p = N$ in which U_1 and U_2 can be diagonalized simultaneously. We seek the vacuum state satisfying

$$U_j \Psi_{\alpha_1 \alpha_2} = e^{i\epsilon^{jk}\alpha_k} \Psi_{\alpha_1 \alpha_2} . \tag{3.1}$$

Define $u(\theta_1) = \langle \theta_1 | \Psi \rangle$. It is found that a linearly independent basis can be taken as

$$u_a(\theta_1) = h\left[\theta_1; \frac{1}{N}(\alpha_1 + 2\pi a), \alpha_2\right] \qquad (a = 0, 1, \cdots, N-1), \tag{3.2}$$

where

$$h[\theta; \alpha, \beta] = e^{i\beta\theta/2\pi} \delta_{2\pi}[\theta - \alpha] . \tag{3.3}$$

We shall denote the a-th vacuum by $|0_a\rangle$ so that $u_a(\theta_1) \equiv \langle \theta_1 | 0_a \rangle$.

The important point to note here is that the vacuum in the gauge field sector is N-fold degenerate. It is this N-fold degeneracy that leads naturally to multi-component many-body wave functions. In the model under consideration, the number of components of the wave function is precisely equal to N.

The wave functions (3.2) were first given in [2]. It has been argued in the literature [4] that the requirement of the modular invariance in the θ_1, θ_2 space further gives a restriction on α_j. So long as the consisency of the theory is concerned, however, α_j's are free parameters of the theory. We mention here that the spectrum of particles depends on α_j's.

Action of W_j on $|0_a\rangle$ can be evaluated easily

$$\begin{aligned} W_1^{\pm 1} |0_a\rangle &= e^{\pm i\theta_1} |0_a\rangle = e^{\pm i(\alpha_1 + 2\pi a)/N} |0_a\rangle , \\ W_2^{\pm 1} |0_a\rangle &= e^{\pm i\theta_2} |0_a\rangle = e^{\pm i\alpha_2/N} |0_{a\mp 1}\rangle . \end{aligned} \tag{3.4}$$

Let us introduce $V_j = e^{-i\alpha_j/N} \hat{V}_j$ where

$$\hat{V}_1 = \begin{pmatrix} 1 & & & \\ & c & & \\ & & c^2 & \\ & & & \ddots \\ & & & & c^{N-1} \end{pmatrix}, \qquad \hat{V}_2 = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}, \tag{3.5}$$

and $c=e^{-2\pi i/N}$. V_j's satisfy

$$V_1V_2 = e^{-2\pi i/N}\, V_2V_1. \tag{3.6}$$

The action of W_j's is summarized as

$$\begin{aligned} \langle 0_a | e^{-i\theta_1} &= (V_1)_{ab} \langle 0_b | \,, \\ \langle 0_a | e^{-i\theta_2} &= (V_2)_{ab} \langle 0_b | \,. \end{aligned} \tag{3.7}$$

A similar structure of the vacuum arises when N is an inverse integer, as is evident from the duality argument above. The form of the matrices V_1 and V_2 is a consequence of the Weyl-Heisenberg algebra, as has been noticed by many authors [12,13].

V. MANY-BODY SCHRÖDINGER WAVE FUNCTIONS

We now define the first quantized many-body Schrödinger wave function from the field operators. The wave function is required to be invariant under large gauge transformations (2.13). The Schrodinger wave function for a q-particle state is the matrix element of q field operators between the vacuum and a given q-particle state. It is important to recognize that the particle (anyon) number q is not arbitrary in the Chern-Simons theory on a torus. As shown in Eq. (2.14), for a neutral anyon system with $\kappa=N$ we have $q=-(N/2\pi)\Phi = -mN$. In other words, the anyon number q must be a multiple of N. Otherwise the theory cannot be consistently formulated.

Since the vacuum is N-fold degenerate, the wave function must have N components. One candidate is

$$\begin{aligned} \phi^{\mathrm{f}}_a(t;\boldsymbol{r}_1,\cdots,\boldsymbol{r}_q) &= \langle 0_a | \Omega \cdot \psi(t,\boldsymbol{r}_1)\cdots\psi(t,\boldsymbol{r}_q) | \Psi_q \rangle \qquad (a=0,\cdots,N-1) \\ \Omega &= \exp\left\{ -i \sum_{j=1}^{q} \left(\frac{x_j\theta_1}{L_1} + \frac{y_j\theta_2}{L_2} \right) \right\} \equiv e^{-i\omega} \,. \end{aligned} \tag{4.1}$$

The gauge invariance requires the presence of θ_j in the definition of the Schrödinger wave function. By construction ϕ^{f} is anti-symmetric under the interchange of two coordinates: $\phi^{\mathrm{f}}_a(\boldsymbol{r}_j \leftrightarrow \boldsymbol{r}_k) = -\phi^{\mathrm{f}}_a$. It is the wave function in the fermion representation.

It should be noticed that one needs an extra index a to specify many-body wave functions. This index, which labels the component of the wave function, is not associated with the individual particle, but rather with the wave function as a whole. It is called a "sheet" index in Ref. [8] and the topological order in Ref. [13]. We have demonstrated here that such index arises from the degenerate nature of the gauge field vacuum.

VI. NON-ABELIAN BRAID GROUP REPRESENTATION

$\phi^{\rm f}$ satisfy the algebra of the braid group on a torus in a rather simple way. Non-trivial factors come in under translations. In accordance with Einarsson's notation [6], we introduce three kinds of basic operations on a system of q identical particles on a torus; (1) σ_j: the interchange of the j-th and $j+1$-th particles, (2) τ_j and ρ_j: translation of the j-th particle along a closed non-contractible loop in the x- and y-direction. In our regular representations of Schrödinger wave functions there is no distinction between clockwise and counter-clockwise n-rotations of two identical particles.

Action of these operations on $\phi^{\rm f}$ is defined by

$$\begin{aligned}
&\sigma_j\phi^{\rm f}(t;\cdots,\boldsymbol{r}_j,\boldsymbol{r}_{j+1},\cdots)=\phi^{\rm f}(t;\cdots,\boldsymbol{r}_{j+1},\boldsymbol{r}_j,\cdots)\,,\\
&\tau_j\phi^{\rm f}(t;\cdots,\boldsymbol{r}_j,\cdots)=\phi^{\rm f}(t;\cdots,T_1\boldsymbol{r}_j,\cdots)\,,\\
&\rho_j\phi^{\rm f}(t;\cdots,\boldsymbol{r}_j,\cdots)=\phi^{\rm f}(t;\cdots,T_2\boldsymbol{r}_j,\cdots)\,.
\end{aligned}\tag{5.1}$$

With the wavefunction $\phi^{\rm f}$ defined by (4.1), we have

$$\sigma_j\phi^{\rm f}(t;\cdots)=-\phi^{\rm f}(t;\cdots)\,.\tag{5.2}$$

$\phi^{\rm f}$ picks up -1 just as fermions. Non-trivial factors come in under the action τ_j or ρ_j. To see how $\phi^{\rm f}_a$ transforms upon loop translations, we evaluate τ_j and ρ_j:

$$\begin{aligned}
\tau_l\phi^{\rm f}_a &= \phi^{\rm f}_a(\cdots T_1\boldsymbol{r}_l\cdots)\\
&= \langle 0_a|e^{-i\theta_1-i\omega}\cdots\psi(t,T_1\boldsymbol{r}_l)\cdots)|\Psi_q\rangle\\
&= e^{-i\beta_1(r_l)+(\pi i/NL_2)\sum_{j=1}^q y_j}\langle 0_a|e^{-i\theta_1}\Omega\cdots|\Psi_q\rangle\\
&= e^{-i\beta_1(r_l)+(\pi i/NL_2)\sum_{j=1}^q y_j}\cdot(V_1)_{ab}\phi^{\rm f}_b\,,
\end{aligned}\tag{5.3}$$

where (3.7) has been made use of. Similarly

$$\begin{aligned}
\rho_l\phi^{\rm f}_a &= \langle 0_a|e^{-i\theta_2-i\omega}\cdots\psi(t,T_2\boldsymbol{r}_l)\cdots)|\Psi_q\rangle\\
&= e^{-i\beta_2(r_l)-(\pi i/NL_1)\sum_{j=1}^q x_j}\cdot(V_2)_{ab}\phi^{\rm f}_b\,.
\end{aligned}\tag{5.4}$$

To summarize

$$\begin{aligned}
\tau_l\phi^{\rm f}_a &= \exp\left(-i\beta_1(r_l)+\frac{\pi i}{N}\sum_{j=1}^q\frac{y_j}{L_2}\right)\cdot(V_1)_{ab}\phi^{\rm f}_b\,,\\
\rho_l\phi^{\rm f}_a &= \exp\left(-i\beta_2(r_l)-\frac{\pi i}{N}\sum_{j=1}^q\frac{x_j}{L_1}\right)\cdot(V_2)_{ab}\phi^{\rm f}_b\,.
\end{aligned}\tag{5.5}$$

With $\beta_j(\boldsymbol{r})$ in (2.12) substituted, (5.5) can be written as

$$\begin{aligned}\tau_l\phi^{\rm f} &= \exp\left(+\frac{\pi i}{N}\sum_{j\neq l}\frac{y_{jl}}{L_2}\right)\cdot V_1\phi^{\rm f}\,,\\ \rho_l\phi^{\rm f} &= \exp\left(-\frac{\pi i}{N}\sum_{j\neq l}\frac{x_{jl}}{L_1}\right)\cdot V_2\phi^{\rm f}\,,\end{aligned}\tag{5.6}$$

where $x_{jl} = x_j - x_l$ etc. Further we understand the following rules:

$$\begin{aligned}\sigma_i[f(\cdots)\phi^{\rm f}(\cdots)] &= f(\boldsymbol{r}_i\leftrightarrow\boldsymbol{r}_{i+1})\sigma_i\phi^{\rm f}(\cdots)\,,\\ \tau_i[f(\cdots)\phi^{\rm f}(\cdots)] &= f(\cdots T_1\boldsymbol{r}_i\cdots)\tau_i\phi^{\rm f}(\cdots)\,,\\ \rho_i[f(\cdots)\phi^{\rm f}(\cdots)] &= f(\cdots T_2\boldsymbol{r}_i\cdots)\rho_i\phi^{\rm f}(\cdots)\,.\end{aligned}\tag{5.7}$$

It is now straightforward to check that the representation given by (5.2) to (5.5) satisfies the braid group algebra given in [6] and [15] (see [9] for details).

VII. SINGULAR GAUGE TRANSFORMATION

The Schrödinger wave function $\phi^{\rm f}$, defined on the covering space of the torus, is a regular function of the coordinates of particles, $\boldsymbol{r}_j$'s. In Einarsson's investigation of fractional statistics on a torus, it is implicit that the Schrödinger wave function, denoted by $\phi^{\rm E}$, is a singular function of $\boldsymbol{r}_j$'s, since it picks up a factor $e^{\pm i\theta_s}$ under the interchange of two identical particles. In [9] we show that Einarsson's and our descriptions are equivalent, $\phi^{\rm E}$ and $\phi^{\rm f}$ being related to each other by a singular gauge transformation.

In fact, we have

$$\phi^{\rm E}_a = \Omega_{\rm sing}\phi^{\rm f}_a\,,\tag{6.1}$$

where the gauge transformation is generated by

$$\begin{aligned}\Omega_{\rm sing} &= \prod_{j\neq k}\left(\frac{\vartheta_1(w_{jk})}{\overline{\vartheta}_1(\overline{w}_{jk})}e^{(\pi L_1/2L_2)(w_{jk}^2-\bar{w}_{jk}^2)}\right)^{1/4N}\\ &= \prod_{j<k}\left(\frac{\vartheta_1(w_{jk})}{\overline{\vartheta}_1(\bar{w}_{jk})}\right)^{1/2N}\times e^{i\pi x_{jk}y_{jk}/NL_1L_2}\,.\end{aligned}\tag{6.2}$$

It can be shown that $\phi^{\rm E}_a$ satisfy free Schrödinger equation. $\Omega_{\rm sing}$ is not well defined at $w_{jk} = 0$, which, however, causes no problem since $\phi^{\rm f}$ vanishes at $w_{jk} = 0$. Accordingly one has to impose a condition

$$\phi^{\rm E}_a = 0 \qquad \text{for} \quad \boldsymbol{r}_j = \boldsymbol{r}_k\,.\tag{6.3}$$

The regularity condition in the ϕ^{f} gauge yields a hard-core type interaction in Einarsson's gauge. The transformation (6.2) was first given by Randjbar-Daemi *et al.* [5] and by Lechner [14].

If one restricts oneself to the particle configuration satisfies $x_1 < x_2 < \ldots < x_q$ and $y_1 < y_2 < \cdots < y_q$. (such a configuration is implicitly assumed in [6]. See also Ref. [15]), then one finds the representations of the braid group generators to be (with ϕ^{E} as basis):

$$\begin{aligned}
\sigma_l &= -e^{-i\pi/N} , \\
\tau_l &= e^{+i\pi(q+1)/N-2\pi i l/N} \cdot V_1 , \\
\rho_l &= e^{-i\pi(q+1)/N+2\pi i l/N} \cdot V_2 .
\end{aligned} \tag{6.4}$$

This is exactly what Einarsson has obtained. It should be emphasized, however, that our results account for more general particle configurations than the one just given. The general formulae are given in [9].

VIII. SUMMARY

In this talk we present the quantisation of Chern-Simons theory coupled with non-relativistic matter on a torus. We show that the Hamiltonian and the total momenta do not commute as operators, but only commute in the physical Hilbert space. We also demonstrate how braid group structures arise in a Chern-Simons theory with integer Chern-Simons coefficient. We would also like to mention that these results can be easily extended to theory with multiple numbers of Chern-Simons fields [16].

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