

## On q-deformed Wigner-Eckart Theorem

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We derive the q-deformed Wigner-Eckart theorem for the q-analogues of vector operators. Using this theorem, we discuss how the energy spectrum of an atom would be modified in a uniform magnetic field if the angular momentum were to satisfy the deformed  $SU(2)_q$ .

## I. INTRODUCTION

Quantum groups' and their associated quantum algebras are remarkable mathematical structures which play substantial roles in rational conformal field theories<sup>2</sup> and in solvable models of statistical mechanics.<sup>3</sup> One of the simplest and fundamental quantum group is  $SU(2)_q$ , the q-deformation of  $SU(2)$  group. The natural strategy is to develop the algebra  $SU(2)_q$  along the line of the angular momentum algebra  $SU(2)$ . Here, we would like to present a q-analogues of the Wigner-Eckart theorem.

Recalling that the standard defining relations for  $SU(2)_q$  are characterized by three generators,  $J_{\pm}$  and  $J_0$ , which satisfy

$$[J_0, J_{\pm}] = \pm J_{\pm}$$

and

$$[J_+, J_-] = J_+ J_- - J_- J_+ = [2J_0], \quad (1)$$

where the notation  $[ ]$  is defined in terms of deformation parameter  $q$  as

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}}. \quad (2)$$

For a generic  $q$ , the Hilbert space with basis state  $|j, m\rangle$ , where  $j$  takes one of the values 0, 1/2, 1, .., is constructed such that

$$J_{\pm} |j, m\rangle = ([j \mp m][j \pm m + 1])^{1/2} |j, m \pm 1\rangle,$$

$$J_0 |j, m\rangle = m |j, m\rangle$$

and

$$\vec{J} \cdot \vec{J} |j, m\rangle = [j][j+1] |j, m\rangle, \quad (3)$$

where  $\vec{J} \cdot \vec{J}$  is the Casimir operator of  $SU(2)_q$

$$\begin{aligned} \vec{J} \cdot \vec{J} &\equiv [J_z][J_z - 1] + J_+ J_- \\ &= [J_z][J_z + 1] + J_- J_+ . \end{aligned}$$

In the limit  $q \rightarrow 1$ , one reproduces the familiar angular momentum results, since  $[x] \rightarrow x$  as  $q \rightarrow 1$ .

## II. q-DEFORMED WIGNER-ECKART THEOREM

Consider a q-deformed vector operator  $\vec{A}$  with three generators,  $A_+$ ,  $A_0$ , and  $A_-$ , which satisfy

$$[A_{\pm}, J_0] = \mp A_{\pm} \quad (4-a)$$

$$[A_{\pm}, J_{\pm}] = 0 \quad (4-b)$$

$$[A_{\pm}, J_{\mp}] = \pm [2A_0] \quad (4-c)$$

$$[A_0, J_0] = 0 \quad (4-d)$$

$$[A_0, J_{\pm}] = \pm A_{\pm} \quad (4-e)$$

The usual Wigner-Eckart theorem<sup>4</sup> states that in a representation that diagonalizes  $J^2$  and  $J_z$ , the matrix element of  $\vec{A}$  between states of the same  $\vec{J}$  are proportional to the matrix elements of  $\vec{J}$ , and the proportionality constant is independent of the magnetic quantum numbers. We shall prove below that the same theorem holds in the case of quantum groups as well.

To prove this, we take matrix elements of Eq. (4) with respect to states,  $|X j m\rangle$ , in which  $J^2$  and  $J_z$  are diagonal. All the other quantum numbers are represented by  $\lambda$  and we assume that  $\lambda$  commutes with angular momentum operator  $\vec{J}$ . That is, we have

$$J_{\pm} |\lambda j m\rangle = \langle j m \pm 1 | J_{\pm} | j m \rangle |\lambda j m \pm 1\rangle \quad (5-a)$$

$$\langle \lambda j m | J_{\pm} = \langle j m | J_{\pm} | j m \mp 1 \rangle \langle \lambda j m \mp 1 |$$
 (5-b)

$$\begin{aligned} J_0 | \lambda j m \rangle &= \langle j m | J_0 | j m \rangle | \lambda j m \rangle \\ &= m | \lambda j m \rangle \end{aligned}$$
 (5-c)

Now, taking matrix elements of (4-a) and using (5-a), we obtain

$$\mp \langle \lambda' j m' | A_{\pm} | \lambda j m \rangle = (m - m') \langle \lambda' j m' | A_{\pm} | \lambda j m \rangle$$
 (6-a)

Therefore  $m' = m \pm 1$ , otherwise the matrix element vanishes. The matrix elements of  $A$ , are now proportional to those of  $J_{\pm}$ , and the proportionality constant may be written as

$$\frac{\langle \lambda' j m \pm 1 | A_{\pm} | \lambda j m \rangle}{\langle j m \pm 1 | J_{\pm} | j m \rangle}.$$
 (6-b)

Applying the similar argument to (4-b), using (5-a) and (5-b), gives

$$\begin{aligned} \langle \lambda' j m' | A_{\pm} | \lambda j m \pm 1 \rangle &\langle j m \pm 1 | J_{\pm} | j m \rangle \\ &= \langle \lambda' j m' \mp 1 | A_{\pm} | \lambda j m \rangle \langle j m' | J_{\pm} | j m' \mp 1 \rangle \end{aligned}$$
 (6-c)

Finally we have the analog of (6-a),  $m' = m \pm 2$ , and we conclude that

$$\begin{aligned} \langle \lambda' j m \pm 2 | A_{\pm} | \lambda j m \pm 1 \rangle &\langle j m \pm 1 | J_{\pm} | j m \rangle \\ &= \langle \lambda' j m \pm 1 | A_{\pm} | \lambda j m \rangle \langle j m \pm 2 | J_{\pm} | j m \pm 1 \rangle \end{aligned}$$

or

$$\frac{\langle \lambda' j m \pm 2 | A_{\pm} | \lambda j m \pm 1 \rangle}{\langle j m \pm 2 | J_{\pm} | j m \pm 1 \rangle} = \frac{\langle \lambda' j m \pm 1 | A_{\pm} | \lambda j m \rangle}{\langle j m \pm 1 | J_{\pm} | j m \rangle}.$$
 (7)

Thus the proportionality constant (6 - b) is independent of  $m$ , and we call it  $\langle \lambda' j || A || \lambda j \rangle_{\pm}$ . This is the definition of the reduced matrix element

$$\langle \lambda' j m' | A_{\pm} | \lambda j m \rangle = \langle \lambda' j || A || \lambda j \rangle_{\pm} \langle j m' | J_{\pm} | j m \rangle.$$
 (8)

We next obtain a similar relation for  $A$ , and show that  $\langle \lambda' j || A || \lambda j \rangle_{+} = \langle \lambda' j || A || \lambda j \rangle_{-}$ . Matrix element of (4-c) gives, with the aid of (5-a)(5-b) and some algebra,

$$\langle \lambda' j m' | [2A_0] | \lambda j m \rangle = \langle \lambda' j || A || \lambda j \rangle_{\pm} \langle j m' | [2J_0] | j m \rangle.$$
 (9)

Equation (9) indicates that the proportionality coefficient must be the same regardless of whether we take the + or - signs. Hence we obtain as a final result

$$\langle \lambda' j m' | \vec{A} | \lambda j m \rangle = \langle \lambda' j || A || \lambda j \rangle \langle j m' | \vec{J} | j m \rangle \quad (10)$$

which proves the theorem.

### III. ZEEMAN EFFECT

If an external magnetic field acts upon an atom, its energy states change. The shift of the energy levels under the influence of an magnetic field is called the Zeeman effect.<sup>4</sup>

So far a direct physical application of the quantum group is absent. As a simple application of previous discussion of the q-deformed Wigner-Eckart theorem, we consider how the energy spectrum of an atom in a uniform magnetic field would be modified, if the angular momentum were to satisfy the deformed  $SU(2)_q$ .

The interaction Hamiltonian of the electron with a uniform magnetic field  $B$  along the z-axis is given by

$$\begin{aligned} H_{int} &= \frac{eB}{2m_e c} (L_z + 2S_z) \\ &= \frac{eB}{2m_e c} (J_z + S_z) \end{aligned} \quad (11)$$

When the field is sufficiently weak so that the effect of interaction is small, we can consider  $H_{int}$  as a perturbation on the IS  $|L j m\rangle$  basis. Thus, we need to evaluate the diagonal elements  $\langle L S j m | J_z + S_z | L S j m \rangle$ .

According to the q-deformed Wigner-Eckart theorem, we have

$$\langle L S j m | S_z | L S j m \rangle = \langle L S j || S || L S j \rangle \langle j m | J_z | j m \rangle . \quad (12)$$

The reduced matrix element may be calculated by using

$$\begin{aligned} \langle L S j m | \vec{S} \cdot \vec{J} | L S j m \rangle &= \langle L S j || S || L S j \rangle \langle L S j m | \vec{J} \cdot \vec{J} | L S j m \rangle \\ &= \langle L S j || S || L S j \rangle [j][j+1] . \end{aligned} \quad (13)$$

The operator  $\vec{S} \cdot \vec{J}$  may be rewritten as

$$2\vec{S} \cdot \vec{J} = \vec{J} \cdot \vec{J} + \vec{S} \cdot \vec{S} - \vec{L} \cdot \vec{L} . \quad (14)$$

Therefore

$$\langle L S j || S || L S j \rangle = \frac{[j][j+1] + [S][S+1] - [L][L+1]}{2[j][j+1]}$$

and we find

$$\langle L S j m | H_{int} | L S j m \rangle = \frac{eB}{2m_e c} mG$$

where

$$G = 1 + \frac{[j][j+1] + [S][S+1] - [L][L+1]}{2[j][j+1]}$$

is the q-deformed Landé factor.

#### IV. SUMMARY

We have obtained the q-deformed Wigner-Eckart theorem for the q-deformed vector operators and discussed its application to the Zeeman effect. For the scalar operator  $\hat{S}$ , with the defining relation  $[\vec{J}, \hat{S}] = 0$ , the q-deformed Wigner-Eckart theorem still hold, that is

$$\langle \lambda' j' m' | \hat{S} | \lambda j m \rangle = \delta_{jj'} \delta_{mm'} \langle \lambda' j' || \hat{S} || \lambda j \rangle .$$

The reduced matrix element  $\langle \lambda' j' || \hat{S} || \lambda j \rangle$  is independent of the magnetic quantum numbers,  $m$ . We hope to generalize in future this theorem to the cases of tensor operators.

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