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Local and Global Existence of Metrics in Two-dimensional Affine Manifolds

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Necessary and sufficient conditions are obtained that a symmetric connection on a two-dimensional manifold should be the Levi-Civita connection of some metric, both locally and globally.

I. INTRODUCTION

The fundamental theorem or lemma of Riemannian (or pseudo-Riemannian) geometry asserts the existence of a unique symmetric connection compatible with a given metric. At present I am investigating various questions of existence and uniqueness of a metric g compatible with a given symmetric connection. The situation in two dimensions in this regard is rather exceptional and it seems worthwhile to present it separately. This and the more general problem have been considered before;[1,2,3] however, the arguments can be refined considerably and result in several interesting points which, as far as I am aware, have been overlooked. The notation used is reasonably standard. I assume that M^m is a smooth m-dimensional manifold endowed with a smooth symmetric connection V. I shall quickly specialize to the case where m is two.

II. PRELIMINARIES

For the moment the development will be general and there will be no restriction on the dimension m of M. Assume a symmetric connection V is given on M. We are looking for a metric g so that V is its Levi-Civita connection. Regard the compatibility condition

$$\nabla g = 0 \tag{2.1}$$

as a system of partial differential equations for the components of g. By "differentiating and equating mixed partials," one obtains the following well known integrability conditions for (2.1):

$$g(X,-)R(Z,W)Y + g(Y,-)R(Z,W)X = 0 \tag{2.2}$$

where X, Y, Z, W are arbitrary vector fields on M. In (2.2) (g(X,-) is a one-form applied to the vector field which follows it. Let us abbreviate (2.2) simply to

$$g \circ R + (g \circ R)^t = 0 \tag{2.3}$$

Equation (2.3) can now be covariantly differentiated repeatedly leading to a sequence of algebraic conditions, which in obvious notation are

$$\begin{aligned} g \circ \nabla R + (g \circ \nabla R)^t &= 0 \\ g \circ \nabla^2 R + (g \circ \nabla^2 R)^t &= 0 \\ &\vdots \\ g \circ \nabla^k R + (g \circ \nabla^k R)^t &= 0. \end{aligned} \tag{2.4}$$

The linear algebraic conditions derived from (2.3) must stabilize for some positive integer N, in the sense that the conditions at the $(N + 1)^{st}$ stage and hence all higher stages are algebraic consequences of (2.3) and its first N covariant derivatives. As was observed by Eisenhart [1], an obvious condition for the existence of g is that the totality of linear conditions comprised by (2.3) and its first N covariant derivatives should possess at least a one dimensional solution space, spanned by a non-degenerate quadratic form.

Another general remark concerning (2.3) is that since $g \circ R$ is skew-symmetric and g is symmetric, the trace of R must be zero. It is easily seen that tr(R) = 0 is equivalent to the Ricci tensor being symmetric.

III. THE TWO-DIMENSIONAL CASE

Now write out (2.3) in component form for the case m = 2. Using the definition of the Ricci tensor K gives

$$\begin{bmatrix} -K_{12} & K_{11} & 0 \\ -K_{22} & K_{21} - K_{12} & K_{11} \\ 0 & K_{22} & -K_{21} \end{bmatrix} \begin{bmatrix} g_{11} \\ g_{12} \\ g_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{3.1}$$

Since K is symmetric the coefficient matrix in (3.1) has rank two provided the given connection is not flat.

For the remainder of the paper I shall assume that V is not flat in the sense that R is nowhere zero. Accordingly the solution to (3.1) consists of multiples of K and K must be a non-degenerate symmetric bilinear form.

If V admits a compatible metric then we may write

$$g = e^{-f} K \tag{3.2}$$

for some function f. Hence

$$\nabla g = e^{-f}[\nabla K - df \otimes K].$$

Thus g is a compatible metric for V if and only if

$$\nabla K = df \otimes K \tag{3.3}$$

Recall that a tensor T (of any type) is said to be recurrent if there exists a one form θ such that

$$\nabla T = \theta \otimes T. \tag{3.4}$$

In particular, (3.3) states that K is recurrent. An interesting phenomenon happens in this regard.

Lemma 3.1: Any symmetric Ricci tensor **K** on $\mathbf{M}^2$ satisfies the identity

$$\nabla_Z \nabla_W K - \nabla_W \nabla_Z K - \nabla_{[Z,W]} K = 0, \tag{3.5}$$

for arbitrary vector fields Z and W.

Proof: Given two other vector fields X and Y one has identically

$$\begin{aligned}(\nabla_Z \nabla_W K - \nabla_W \nabla_Z K - \nabla_{[Z,W]} K)(X,Y)\\ = K(X,-)R(Z,W)Y + K(Y,-)R(Z,W)X = 0\end{aligned} \tag{3.6}$$

(Compare (2.1) and (2.2)). Now use the identity which holds in $\mathbf{M}^2$ (see [4, p. 110]) relating K and R:

$$R(Z,W)Y = K(Z,Y)W - K(W,Y)Z. \tag{3.7}$$

Substituting (3.7) in the right hand side of (3.6) gives

$$\begin{aligned}&K(X,-)R(Z,W)Y + K(Y,-)R(Z,W)X\\ &= K(X,W)K(Z,Y) - K(X,Z)K(W,Y)\\ &\quad + K(Y,W)K(Z,X) - K(Y,Z)K(W,X)\\ &= 0\end{aligned}$$

due to the symmetry of K. Thus the left-hand side of (3.6) is zero, proving the lemma.

Proposition 3.2: If K is symmetric and satisfies (3.4), then θ is closed.

Proof: The proof is immediate from Lemma 3.1 and the following identity obtained from (3.4) with K replacing **T**:

$$\nabla_X \nabla_Y K - \nabla_Y \nabla_X K - \nabla_{[X,Y]} K = d\theta(X,Y)K. \tag{3.8}$$

The preceding discussion is resumed in the following theorem.

Theorem 3.3: Local necessary and sufficient conditions for a non-flat, symmetric connection V on M^2 to be the Levi-Civita connection of a metric g are that the Ricci tensor of V should be

(i) symmetric and non-degenerate

(ii) recurrent.

Furthermore, if the one-form θ appearing in (3.4) is exact, then g exists globally and is unique.

I conclude this paper with an example of a non-simply connected manifold M^2 which has a symmetric connection that is locally, but not globally, Levi-Civita. Consider first of all R^2 with standard global coordinates (x,y) and metric g given by

$$g = e^{2(x+y^2)}(dx^2 + dy^2) \tag{3.9}$$

It is easy to check that the geodesic flow Γ of g on the tangent bundle TR' with standard coordinates $(x,y,\dot{x},\dot{y})$ is given by

$$\Gamma = \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} - [\dot{x}^2 + 4y\dot{x}\dot{y} - \dot{y}^2]\frac{\partial}{\partial x} - [2y\dot{y}^2 + 2\dot{x}\dot{y} - 2y\dot{x}^2]\frac{\partial}{\partial y}. \tag{3.10}$$

Now consider the translation $(x,y) \rightarrow (x+1,y)$ on R^2. Then clearly Γ is invariant under this translation (or more precisely its lift to TR")) whereas g is not. Hence the Levi-Civita connection of g passes to the quotient space $S^1 \times R$ and is locally but not globally a metric connection.

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