CHINESE JOURNAL OF PHYSICS VOL. 29, NO. 3 JUNE 1991

Space-Time Symmetries, Gauge Fields and Differential Forms

M. K. Fung (馮明光)
Department of Physics, National Taiwan Normal University
Taipei, Taiwan 117, R. O. C.

(Received Jan. 7, 1991; revised manuscript received Feb. 11, 1991)

We discuss space-time symmetries of gauge fields by the tools of differential forms and the homotpy formula. We give examples of applications for the U(1) Maxwell field and the SU(2) gauge field.

I. INTRODUCTION

Symmetry considerations dominate the study of physics. Generically, we have two kinds of symmetries. We have space-time symmetries including translation, rotation, boost, conformal transformation, etc. On the other hand we have internal symmetries, e.g. gauge transformations — abelian and nonabelian. It was found that there is a subtle interplay between space-time symmetries and gauge symmetries. Forgacs and Manton[1] had given a formulation of the problem in terms of infinitesimal coordinate transformations, i.e. using the tools of Lie derivatives. There is a comprehensive review of this subject by Jackiw.[2] Harnad, Shnider and Vinet[3] studied the problem employing fibre bundle langauge. Global analysis had also been made by Gu and Hu.[4] Earlier works include those of Bergmann and Flaherty[5] and Trautman.[6]

Here, we propose to study this problem of space-time symmetries in gauge theories via the method of Lie derivatives. We suggest the use of differential forms in our analysis. The advantages of using differential forms over the use of indexed tensor fields are many.[7] Firstly, differential forms are nice geometrical objects. They possess the geometrical operation of the exterior derivative d. Hence acting on differential forms the Lie derivative takes on a very simple form — the Car-tan homotopy formula.[8] Also, differential forms behave nicely under mappings. The pullback of a differential form is still a differential form under a differentiable mapping. This is not generally true for the derivative mapping of a vector field. So in Sect. II we shall discuss Lie derivatives on differential forms, especially on the potential one-form A. In Sect. III we follow Forgacs and Manton[1] to formulate the symmetry equation in terms of Lie derivatives. We have to consider several symmetries simultaneously. This leads to a consistency equation. Indeed, we solve the consistency equation and the symmetry equation for the U(1) case in Sect. IV and for the nonabelian SU(2) case in Sect. V. In Sect. VI we draw our conclusion.

II. LIE DERIVATIVES AND THE HOMOTOPY FORMULA

Consider a point transformation

$$x^\mu \rightarrow x'^\mu , \tag{2.1}$$

which in infinitesimal form assumes

$$x'^\mu = x^\mu + \epsilon X^\mu \tag{2.2}$$

where ϵ is an infinitesimal. We say that the transfomration is generated by a one-parameter group of diffeomorphism through the vector field X^μ. This also defines, passively, a coordinate transformation. Tensor fields are transformed likewise. For example, for a covariant vector field

$$v'_\mu (x') = v_\alpha (x) \frac{\partial x^\alpha}{\partial x'^\mu} . \tag{2.3}$$

The Lie difference with respect to X^μ is then defined to be

$$\delta^* v_\mu = v_\mu (x) - v'_\mu (x), \tag{2.4}$$

and the Lie derivative is just the limit

$$L_X v_\mu = \lim_{\epsilon \to 0} \frac{v_\mu (x) - v'_\mu (x)}{\epsilon} \tag{2.5}$$

Expanding out the infinitiesimals and taking limit, we get

$$L_X v_\mu = (\partial_\mu X^\rho) v_\rho + X^\rho \partial_\rho v_\mu . \tag{2.6}$$

We have similar formulae for other tensor fields. Observe that the Lie derivative preserves the tensor type since we are taking differential at the same point.

Acting on differential forms, i.e. skew-symmetric covariant tensor fields, the Lie derivative takes an especially simple form

$$L_X = i_X d + d i_X , \tag{2.7}$$

where d is the exterior derivative operator and i_X is the inner derivative which acts on forms lowering their degree by one.

$$i_X \omega(X_1, \ldots, X_{p-1}) = \omega(X, X_1, \ldots, X_{p-1}) \quad . \tag{2.8}$$

It is the usual contraction operation with the vector field X^μ. We shall not give a proof of this formula. Rather, we shall discuss it in the light of the concept of derivation and anti-derivation.*

If a linear operator θ maps a p-form to a (p+r)-form, it is called an operator of degree r. The operator θ is called a derivation if it is of even degree and for every form α, β the Leibnitz rule is satisfield.

$$\theta(\alpha \wedge \beta) = (\theta\alpha) \wedge \beta + \alpha \wedge (\theta\beta) \; . \tag{2.9}$$

It is called an antiderivation if it is of odd degree, and for every form α, β

$$\theta(\alpha \wedge \beta) = (\theta\alpha) \wedge \beta + (-1)^{\deg\alpha} \alpha \wedge (\theta\beta) \; . \tag{2.10}$$

Hence d is an antiderivation of degree +1, while ix is an antiderivation of degree -1. If θ_1, θ_2 are two antiderivations $(\theta_1\theta_2 + \theta_2\theta_1)$ is a derivation. Hence we have

$$L_X = i_X d + d i_X \; , \tag{2.7}$$

where L_X is a derivation of degree 0.

Also the commutator or Lie bracket of two derivations is a derivation and the commutator of a derivation and an antiderivation is an antiderivation where the commutator is an antiderivation where the commutator is defined as

$$[\theta_1, \theta_2] = \theta_1\theta_2 - \theta_2\theta_1 \; . \tag{2.11}$$

The following relations are useful for manipulation and can be understood in the above context.

$$[L_X, L_Y] = L_{[X,Y]} \tag{2.12a}$$

$$[L_X, i_Y] = i_{[X,Y]} \tag{2.12b}$$

III. SYMMETRY EQUATION FOR THE GAUGE FIELDS

Now we are going to discuss the space-time symmetries of the gauge fields. The space-time symmetries are described by the Lie group action, e.g. 3-dimensional rotation by the SO(3) group. The group action consists of a set of one parameter group of diffeomorphisms described by the vector fields X^μ, which are closed under the action of the commutator bracket. We use the formulation of differential forms for the gauge fields. To be general we give formulae for the nonabelian case.

Suppose that the gauge group G has generators T^a with commutation relations

$$[T^a, T^b] = f^{abc} T^c \tag{3.1}$$

and normalisation

$$\mathrm{Tr}(T^a T^b) = -\tfrac{1}{2}\delta^{ab} , \tag{3.2}$$

the gauge potential can be written as

$$A_\mu = A^a_\mu T^a \tag{3.3}$$

The gauge potential gives rise to a matrix-valued one-form

$$A(x) = A_\mu (x) dx^\mu \tag{3.4}$$

The field strength is a two-form,

$$F = dA + A^2 . \tag{3.5}$$

A gauge transformation acting on A takes the imhomogeneous form

$$\begin{aligned} A' &= g^{-1} A g + g^{-1} dg , \qquad g \epsilon G \\ &= A + g^{-1} Dg , \end{aligned} \tag{3.6}$$

where D is the covariant derivative on the matrix,

$$Dg = dg + [A, g] , \tag{3.7}$$

and F transforms covariantly

$$F' = g^{-1} F g , \qquad g \epsilon G . \tag{3.8}$$

Infinitesimaly, if $g = 1 + \epsilon W$, then Eq. (3.6), (3.8) take the form

$$A' = A + \epsilon DW \tag{3.6a}$$

$$F' = F + \epsilon [F, W] \quad . \tag{3.8a}$$

Now if the gauge potential A is invariant under a point transformation generated by the vector field X^μ, its Lie derivative must be zero up to a guage transformation.

$$L_X A = DW_X \quad . \tag{3.9}$$

This is the symmetry equation. If we require gauge invariance of the symmetry equation, then under a gauge transformation $g \in G$, W_X transforms as

$$W'_X = g^{-1} W_X g + g^{-1} L_X g \quad . \tag{3.10}$$

It is important to observe that for a single symmetry, it is always possible to reduce W_X to zero. The theory becomes interesting for the case when there are several symmetries so that the W_X functions cannot be reduced to zero simultaneously. We should remark that all formulae in Refs. [1] and [2] following from the symmetry Eq. (3.9) can be derived simply by using the homotopy formula (2.7) together with the formulae (2.12a, b). We shall not elaborate along this line in what follows.

For the case of more than one symmetries, we have the consistency condition

$$(L_X L_Y - L_Y L_X)A = L_{[X, Y]} A \quad , \tag{3.11}$$

because the commutator of two vector fields is still a vector field. This reduces after some manipulations to

$$L_X W_Y - L_Y W_X - [W_X, W_Y] = W_{[X, Y]} \tag{3.12}$$

This will be called the consistency equation. Surprisingly this consistency equation places a severe constraint on the form of the W_X functions which we shall see in the following section.

IV. THE U(1) MAXWELLFIELD

We now apply the theory to construct spherically symmetric potentials of the U(1) Maxwell field. The theory is simple because the field is abelian. Hence the symmetry equation reduces to

$$L_X A = dW_X \quad , \tag{4.1}$$

where A is a scalar one-form and W_X is a scalar function. The consistency equation is then written as

$$L_X W_Y - L_Y W_X = W_{[X,Y]} \tag{4.2}$$

For the rotational group SO(3), we take the rotations at the x, y, z axes as our one-parameter vector fields.

$$\begin{aligned} X &= y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \\ Y &= z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \\ Z &= x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \quad . \end{aligned} \tag{4.3}$$

with the commutation rule $[X, Y] = -Z$, and cyclic permutation.

We shall work in spherical coordinates, where the three vector fields take the form

$$\begin{aligned} X &= -\sin\phi\frac{\partial}{\partial\theta} - \cot\theta\cos\phi\frac{\partial}{\partial\phi} \\ Y &= \cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi} \\ Z &= \frac{\partial}{\partial\phi} \end{aligned} \tag{4.4}$$

We can first choose W_Z to be zero. Hence the consistency equations become

$$L_X W_Y = L_Y W_X \,, \tag{4.5}$$

and

$$\begin{aligned} L_Z W_X &= -W_Y \\ L_Z W_Y &= W_X \quad . \end{aligned} \tag{4.6}$$

Obviously one solution is $W_X = W_Y = 0$ which gives the explicitly spherically symmetric field. Now suppose W_X, W_Y depending only on θ and ϕ, the two-sphere which is the orbit Of SO(3), we get

$$\begin{aligned} W_X &= f(\theta)\cos\phi \\ W_Y &= f(\theta)\sin\phi \quad . \end{aligned} \tag{4.7}$$

Substitute these into Eq. (4.5) we get a differential equation for $f(\theta)$

$$\frac{df}{d\theta} + \cot\theta\, f = 0 \quad , \tag{4.8}$$

which has solution

$$f(\theta) = \frac{im}{\sin\theta} \tag{4.9}$$

where m is a constant. We can now study the symmetry equation with the above values of W_X, W_Y and W_Z We take the ansatz,

$$A = A_\theta(\theta, \phi)d\theta + A_\phi(\theta, \phi)d\phi \quad . \tag{4.10}$$

Imposing $L_Z A = 0$, we get

$$\frac{\partial A_\theta}{\partial\theta} = \frac{\partial A_\phi}{\partial\phi} = 0 \quad . \tag{4.11}$$

From eq. (4.6) we get

$$A_\theta = 0 \quad \text{and} \quad A_\phi = -im\cos\theta \, . \tag{4.12}$$

Hence our potential is

$$A = -im\cos\theta \quad d\phi \tag{4.13}$$

We know that on a two-sphere the potential A must have singularities. In our choice, the singularities occur at $\theta = 0$ and π. These singularities can be removed by gauge transformations, $e^{im\phi}$ at $\theta = 0$ and $e^{-im\phi}$ at $\theta = \pi$. A global analysis gives us the quantisation rule

$$e^{i4\pi m} = 1 \tag{4.14}$$

where m are half integers. This is the Dirac rule of quantisation and the spherically symmetric field is a Dirac monopole field.

V. SU(2) GAUGE FIELD

We now consider 3-dimensional rotational symmetric SU(2) gauge field. To begin with, we examine the consistency equation

$$L_X W_Y - L_Y W_X - [W_X, W_Y] = W_{[X,Y]} \tag{5.1}$$

We notice that, in particular, one solution is the constant solution where the W's generate a SO(3) subgroup of the gauge group. This aspect had been extensively studied in Ref. [9]. We shall skip this part.

We now take the choice $W_Z = 0$. The consistency equation becomes

$$L_X W_Y - L_Y W_X = [W_X, W_Y] \tag{5.2}$$

and

$$L_Z W_X = -W_Y \, ,$$

$$L_Z W_Y = W_X \quad . \tag{5.3}$$

The solution is simply

$$W_X = F(\theta)\cos\phi$$

$$W_Y = F(\theta)\sin\phi \tag{5.4}$$

with

$$F(\theta) = \Omega/\sin\theta \tag{5.5}$$

where Ω is a constant matrix. We shall see later that Ω is not arbitrary but has to satisfy a constraint. Essentially we can take $\Omega = m(\frac{1}{2} i\sigma_3)$ where m is an integer.

The case $\Omega = 0$ gives the explicitly symmetrical field. For nonvanishing Ω, we can take

$$A = A_0 dt + A_r dr + A_\theta d\theta + A_\phi d\phi \quad . \tag{5.6}$$

From the symmetry equation we can see immediately that A_0, A_r must be independent of θ and ϕ and must commute with Ω. We still have two possibilities. We can take A commuting with Ω. This reduces essentially to the abelian Dirac monopole.

$$A = A_0(r, t)dt + A_r(r, t)dr - \Omega\cos\theta d\phi \quad . \tag{5.7}$$

The other case is for A_0, A_r not commuting with Ω. We can take

$$A_\phi = -\Omega\cos\theta + A_\phi' \quad . \tag{5.8}$$

Because of the extra $[A, W_X]$ term we can get from the symmetry equation

$$(i_X d + d i_X)(A_\theta\, d\theta + A'_\phi\, d\phi) = [A_\theta\, d\theta + A'_\phi\, d\phi, W_X] \tag{5.9}$$

the following equations

$$-\sin\phi \frac{\partial A'_\phi}{\partial\theta} + \cot\theta \sin\phi A'_\phi - \cos\phi A_\theta = [A'_\phi, \frac{\Omega}{\sin\theta} \cos\phi] \tag{5.10}$$

and

$$-\sin\phi \frac{\partial A_\theta}{\partial\theta} + \frac{\cos\phi}{\sin^2\theta} A'_\phi = [A_\theta, \frac{\Omega}{\sin\theta}\cos\phi] . \tag{5.11}$$

The equations are consistently satisfied if we can make them independent of ϕ. Hence we choose

$$A'_\phi = \sin\theta\,\Omega_\phi , \qquad A_\theta = \Omega_\theta , \tag{5.12}$$

where $\Omega_\theta, \Omega_\phi$ are constant matrices satisfying

$$[\Omega_\phi, \Omega] = -\Omega_\theta$$

$$[\Omega_\theta, \Omega] = \Omega_\phi \tag{5.13}$$

Again, to interpret the results in terms of overlapping coordinate patches we require

$$e^{4\pi\Omega} = 1 \tag{5.14}$$

In particular we can choose $\Omega = \frac{1}{2} i\sigma_3$, then

$$\Omega_\theta = \Phi_1 (\tfrac{1}{2} i\sigma_1) - \Phi_2 (\tfrac{1}{2} i\sigma_2)$$

$$\Omega_\phi = \Phi_2 (\tfrac{1}{2} i\sigma_1) + \Phi_1 (\tfrac{1}{2} i\sigma_2) \tag{5.15}$$

where Φ_1, Φ_2 Witten's rary fansatz.[10,1,2] and t. This is the famous

VI. CONCLUSION

We have discussed the space-time symmetries of gauge fields using differential forms. We have given application of the theory to U(1) Maxwell field and SU(2) gauge field. Our derivations are along similar lines as Jackiw[2] since we solve the consistency equation directly. Our method differs somewhat from that of Forgacs and Manton[1] who use coset space techniques. We hope that our derivations are simpler by comparison.

Of course we have not fully employed the geometrical tools of differential forms and we think that there are still many geometrical meanings behind our manipulations. Our method can readily be applied to other cases. One immediate case would be to examine the SO(5) symmetry of Yang's SU(2) monopole.[11] We think that to extend our classical results to the quantised theory would be of equal importance.

ACKNOWLEDGEMENT

This work is supported in part by the National Science Council, R.O.C.

REFERENCES

1. P. Forgacs and N. S. Manton, Comm. Math. Phys. 72, 15 (1980).
2. R. Jackiw, Acta Phys. Austr., Suppl. XXII 383 (1980).
3. J. Harnad, S. Shnider and L. Vinet, J. Math. Phys. 21, 2719 (1980).
4. Gu Chaohao and Hu Hesheng, Comm. Math. Phys. 79, 75 (1981).
5. P. Bergmann and E. Flaherty, J. Math Phys. 19, 212 (1978).
6. A. Trautman, Bull. Acad. Polon., Ser. Sci. Phys. et Astron. 27, 7 (1979).
7. H. Flanders, ***Differential Forms with Applications for the Physical Sciences*** (Academic Press, New York 1963).
8. A. Lichnerowicz, ***Geometry of Groups of Transformations*** (Noordhoff International Publishing, 1977).
9. P. Goddard and D. Olive, Rep. Prog. Phys. 41, 1367 (1978).
10. E. Witten, Phys. Rev. Lett., 38, 121 (1977).
11. C. N. Yang, J. Math. Phys. 19, 320 (1978).