

## Higher Derivative Interactions in the Interface and Surface Model

Simon C. Lin (林誠謙)

*Institute of Physics, Academia Sinica, Nankang, Taiwan*

(Received June 8, 1987)

A conventional field theoretical treatment of geometrically invariant interactions in the interface model is presented. This treatment includes the surface area term as well as higher derivative interactions. The result of our calculation is consistent with our effective potential treatment published earlier and should also be able to confirm the validity of the epsilon expansion in the interface model.

## I. INTRODUCTION

Higher derivative interactions in the interface and surface model' include surface area, curvature and contraction of the first and second fundamental forms, etc.. These geometrically invariant interactions are generally complicated to deal with, and are usually not renormalisable in the higher dimensions. However, recent interest in low dimensional interface- and membrane-type physics<sup>2</sup> has generated a series of investigations in the renormalisation group behaviour of these strange geometric interactions<sup>3</sup>. Unconventional results of the field theoretical calculations have been reported<sup>3,4</sup>, these include the infrared and ultraviolet behaviour of different types of interactions', a new way to define the renormalisation constant in certain types of interactions<sup>5</sup>, potential<sup>7</sup>,

Due to the recent confusion about validity of the epsilon expansion in the interface model<sup>8</sup> and in response to the study of the naively marginal perturbations<sup>9</sup>, in this paper we shall extend the original two- and four-point perturbative calculations of the surface area (or surface tension) term in the interface model given by Wallace and Zia<sup>10</sup> to eight-point vertex functions in order that systematic treatment of other higher derivative interactions would be more comprehensible. This is done carefully to avoid employing the techniques of the extended effective potential method" which we have improved before. By the same explicit method presented here for the interface model, perturbations to the interface model could be fitted naturally into the same context as the

surface tension itself. This will be treated in the final section.

## II. PERTURBATIVE CALCULATIONS OF INTERFACE MODEL

In their treatment of the statistical mechanics for the interface between two discrete thermodynamic phases, Wallace and Zia<sup>10</sup> (WZ) introduced a field "  $f$  " as the dynamical variable to describe the deviation from planar of an essentially sharp interface. The reduced Hamiltonian is given by,

$$H = \frac{1}{T} \int d^{d-1} y (\sqrt{1 + (\nabla f)^2} + \frac{1}{2} m^2 f^2) \quad (1)$$

where the first term is the surface area of the interface and the second term is the pinning potential, which becomes gravity in the case of a liquid-gas system. Although equation (1) is a phenomenologically sound Hamiltonian, it is in fact a derived effective Hamiltonian of the generic Ginsburg-Landau-Wilson Hamiltonian for critical phenomena<sup>12</sup>.

By writing the partition function  $Z$  as

$$Z = \int D[f] e^{-H}, \quad (2)$$

the perturbation theory can be set up as

$$Z = \int D[f] e^{-H_0} [1 + (-H_1) + (-H_1)^2 + \dots], \quad (3)$$

with the expanded Hamiltonian written in the form

$$H = \frac{1}{T} \int d^{d-1} y \left\{ 1 + \frac{1}{2} m^2 f^2 + \frac{1}{2} (\nabla f)^2 - \frac{1}{8} [(\nabla f)^2]^2 + \frac{1}{16} [(\nabla f)^2]^3 + \dots \right\}. \quad (4)$$

where  $H_0 = \int d^{d-1} y [\frac{1}{2} (\nabla f)^2 + \frac{1}{2} m^2 f^2]$ ;  $H_1$  represents the rest of the Hamiltonian. It is inevitable to introduce an infinite series of interactions in full perturbation theory, because of the square root term of surface tension. Fortunately, by the aid of renormalisation group techniques<sup>13</sup>, it is possible to extract useful information from the lower order (up to two loops in WZ) calculations without solving the complete perturbation theory. For our purpose of demonstration, it is sufficient to consider the following interactions,

$$\begin{aligned} \text{Four-point vertex: } & \frac{1}{8T} [(\nabla f)^2]^2 \\ \text{Six-point vertex: } & \frac{-1}{16T} [(\nabla f)^2]^3 \\ \text{Eight-point vertex: } & \frac{5}{128T} [(\nabla f)^2]^4 \end{aligned} \quad (5)$$

$$\text{Ten-point vertex: } -\frac{7}{256T} [(\nabla f)^2]^5 .$$

The propagator is simply

$$\frac{T}{q^2 + m^2} \quad (6)$$

Since there are extra momenta, compared to the ordinary scalar field theory, associated with vertices, it is more elegant to introduce symmetric tensors as part of the vertices. We shall write Feynman's rule for the theory as,

Four-point vertex:

$$\begin{array}{c} \diagup \quad \diagdown \\ i \quad \quad \quad \ell \\ \diagdown \quad \diagup \\ j \quad \quad \quad k \end{array} \quad \frac{1}{8T} S_{ijkl} q_i q_j q_k q_\ell$$

Six-point vertex:

$$\begin{array}{c} \diagup \quad \diagdown \\ i \quad \quad \quad n \quad \quad \quad m \\ \diagdown \quad \diagup \\ j \quad \quad \quad k \end{array} \quad -\frac{1}{16T} S_{ijklmn} q_i q_j q_k q_\ell q_m q_n \quad (7)$$

etc..

The fourth order and sixth order symmetric tensors are given by

$$S_{ijkl} = \frac{1}{3} (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}),$$

and (8)

$$S_{ijklmn} = \frac{1}{15} (\delta_{ij} \delta_{kl} \delta_{mn} + 14 \text{ terms}) .$$

The eighth order and higher symmetric tensors are constructed similarly,

$$S_{ijkpmnop} = \frac{1}{105} (\delta_{ij} \delta_{kl} \delta_{mn} \delta_{op} + 104 \text{ terms}) . \quad (9)$$

The use of symmetric tensors is very similar in construction to the  $O(n)$  theory<sup>13</sup>, with only a subtle difference of the counting symmetry factor for duplicated diagrams. The advantage of introducing such a seemingly complicated formalism lies in the easy manipulation of higher-loop dimensionally regularised loop integrals, which could be an extremely difficult task if the formalism was not set up properly. Formulae for symmetric tensor manipulation are given at the end of this article as an appendix. The formalism here has been prepared in such a way that the Feynman's diagram calculations for higher derivative interactions are readily prepared for algebraic manipulation on computers.

Let us summarise the result of two- to eight-point vertex functions up to one loop:

(I) Two-point function,

$$\Gamma^{(2)} = \text{---} - \text{---} \circ \text{---}$$

$$\text{where } \text{---} \text{---} = \frac{1}{T} (q^2 + m^2), \quad (10)$$

and

$$\begin{aligned} \text{---} \text{---} \text{---} \text{---} &= 12 q_i q_j \left( \frac{1}{8T} \right) S_{ijkl} \cdot T \cdot \int \frac{1}{(2\pi)^{d-1}} \frac{k_l k_k d^{d-1} k}{k^2 + m^2} \\ &\cong -q^2 \frac{1}{\epsilon} m^\epsilon, \end{aligned} \quad (11)$$

where  $\epsilon = d - 1$ .

This is carried out by standard techniques<sup>14</sup> of dimensional regularisation, symmetric tensor contraction,  $\epsilon$ -expansion and the minimal subtraction scheme. The total two-point function should read

$$\Gamma^{(2)}(q) = \frac{q^2}{T} \left[ 1 + \frac{T}{\epsilon} m^\epsilon + O(T^2) \right] + \frac{m^2}{T}. \quad (12)$$

(II) Four-point function,

$$\text{---} \text{---} \text{---} \text{---} = \frac{4 \cdot 3 \cdot 2}{8T} S_{ijkl} q_i q_j q_k q_l = \frac{3}{T} q^4 \quad (13)$$

$$\begin{aligned} \text{---} \text{---} \text{---} \text{---} &= -\frac{6 \cdot 5 \cdot 4 \cdot 3}{16T} q_i q_j q_k q_l S_{ijklmn} \\ &\times \frac{T}{(2\pi)^{d-1}} \int \frac{k_m k_n d^{d-1} k}{k^2 + m^2} \\ &\cong q^4 \frac{18}{\epsilon} m^\epsilon \end{aligned} \quad (14)$$

$$\text{---} \text{---} \text{---} \text{---} = \frac{(4!)^2}{2! 8^2} q_i q_j q_k q_l S_{ijmn} S_{stkl}$$

$$\begin{aligned}
& \times \int \frac{k_m k_n k_s k_t d^{d-1} k}{(k^2 + m^2) [k - (q_1 + q_2)]^2 + m^2} \\
& + (2 \text{ permutations}) \\
& \cong -3 \cdot \frac{5}{\epsilon} q^4 m^\epsilon .
\end{aligned} \tag{15}$$

The last diagram employs the techniques of Feynman parametrisation. The final result is

$$\Gamma^{(4)}(q) = \frac{3}{T} q^4 \left[ 1 + \frac{T}{\epsilon} m^\epsilon + 0(T^2) \right] \tag{16}$$

(III) Six-point function,

$$\begin{aligned}
& \text{Diagram: A central vertex with six external lines (two horizontal, two diagonal, two vertical) and a loop connecting two of the diagonal lines.} \\
& = \frac{1}{16T} S_{ijklmno} q_i q_j q_k q_l q_m q_n \\
& = -\frac{3 \cdot 5 \cdot 3}{T} q^6 = -\frac{3 \cdot 15}{T} q^6
\end{aligned} \tag{17}$$

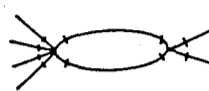
$$\begin{aligned}
& \text{Diagram: A central vertex with six external lines (two horizontal, two diagonal, two vertical) and a loop connecting two of the diagonal lines, with an additional loop connecting two of the vertical lines.} \\
& = \frac{5}{128T} (8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3) q_i q_j q_k q_l q_m q_n \\
& \times S_{ijklmnop} \times \frac{T}{(2\pi)^{d-1}} \int \frac{k_o k_p d^{d-1} k}{k^2 + m^2} \\
& = \frac{5}{128} (8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3) q_i q_j q_k q_l q_m q_n S_{ijklmnop} \\
& \times \frac{1}{(2\pi)^{d-1}} \frac{1}{d-1} \int \frac{k^2 d^{d-1} k}{k^2 + m^2} \\
& = \frac{5}{128} (8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3) \frac{1}{7} [(d-1) + 6] q_i q_j q_k q_l q_m q_n S_{ijklmno} \\
& \times \frac{1}{(d-1)} \frac{1}{(2\pi)^{d-1}} \int \frac{k^2 d^{d-1} k}{k^2 + m^2} \\
& \cong \frac{-5 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{128} \frac{6}{7} q^6 \frac{m^\epsilon}{\epsilon} ,
\end{aligned} \tag{18}$$

where the contraction of the eighth-order symmetric tensor is given by

$$S_{ijk\ell mno} = \frac{1}{7} [(d-1) + 6] S_{ijk\ell mn} \quad , \quad (19)$$

the contribution of this diagram reads

$$-(45 \cdot 15) q^6 \frac{m^\epsilon}{\epsilon} \quad . \quad (20)$$



$$= \frac{25 (4!)^2}{2!} \left( \frac{1}{8T} \right) \left( -\frac{1}{16T} \right) q_i q_j q_k q_\ell$$

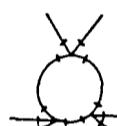
$$\times S_{ijk\ell mn} S_{opqr} q_q q_r$$

$$\times \frac{T^2}{(2\pi)^{d-1}} \int \frac{k_m k_n k_o k_p d^{d-1}k}{(k^2 + m^2) [k - (q_1 + q_2 + q_3 + q_4)]^2 + m^2}$$

A

$$+ \text{ (permutations)}$$

$$\cong (81 \cdot 15) q^6 \frac{m^\epsilon}{\epsilon} \quad . \quad (21)$$



$$= \frac{6 \cdot (4!)^3}{3!} \left( \frac{1}{8T} \right)^3 q_i q_j q_k q_\ell q_m q_n$$

$$\times S_{ijop} S_{k\ell qr} S_{mnst}$$

$$\times \frac{T^3}{(2\pi)^{d-1}} \int \frac{k_o k_p k_q k_r k_s k_t d^{d-1}k}{(k^2 + m^2) \{ [k - (q_1 + q_2)]^2 + m^2 \} \{ [k + (q_3 + q_4)]^2 + m^2 \}}$$

$$+ \text{ (permutations)}$$

$$\cong -(39 \cdot 15) q^6 \frac{m^\epsilon}{\epsilon} \quad . \quad (22)$$

The six-point vertex function is therefore written as

$$\Gamma^{(6)}(q) = \frac{-45}{T} q^6 \left[ 1 + (15 - 27 + 13) \frac{T}{\epsilon} m^\epsilon + O(T^2) \right]$$

$$= -\frac{45}{T} q^6 \left[ 1 + \frac{T}{\epsilon} m^\epsilon + O(T^2) \right] \quad . \quad (23)$$

## (IV) Eight-point function,

The eight-point vertex function up to one loop is graphically given by

$$\begin{aligned}
 \Gamma^{(8)}(q) \cong & \text{[Tree diagram: 8 lines meeting at a central point]} + \text{[Loop diagram: circle with 8 external lines]} + \text{[Loop diagram: circle with 8 external lines, different topology]} \\
 & + \text{[Loop diagram: circle with 8 external lines, another topology]} + \text{[Loop diagram: circle with 8 external lines, another topology]} + \text{[Loop diagram: circle with 8 external lines, another topology]} \\
 \cong & \frac{105 \cdot 15}{T} q^8 \left[ 1 + \frac{T}{\epsilon} m^\epsilon + O(T^2) \right], \quad (25)
 \end{aligned}$$

where the complicated intermediate formulae are omitted. They are calculable in the same mechanical way as presented for four- and six-point vertex functions.

We are now ready to examine the renormalisation group behaviour of the interface model. From dimensional analysis, it is easy to see that

$$[T] = \kappa^{-\epsilon}, \quad [f] = \kappa^{-1} \quad \text{and} \quad [m^2] = \kappa^2, \quad (26)$$

where  $\kappa$  is the momentum scale. The general structure of the singular part of the  $(2n)$ th-order vertex function can be written down rather confidently as

$$\Gamma^{(2n)} \sim \frac{1}{T} \left[ 1 + \frac{T}{\epsilon} m^\epsilon + O(T^2) \right], \quad (27)$$

up to one loop. If we define temperature  $T$  in the following way

$$T \equiv \frac{t_R}{\kappa^\epsilon} \left[ 1 + a(\epsilon) t_R \right], \quad (28)$$

with dimensionless  $t_R$ . The structure of  $a(\epsilon)$  is to be determined in our minimal subtraction scheme. Substitute the above  $T$  into the general vertex function  $\Gamma^{(2n)}$ ,

$$\begin{aligned}
 \Gamma^{(2n)} & \sim \frac{\kappa^\epsilon}{t_R} (1 - a(\epsilon) t_R) \left[ 1 + \frac{t_R}{\epsilon} \left( \frac{m}{\kappa} \right)^\epsilon (1 + a(\epsilon) t_R) \right] \\
 & \sim \frac{\kappa^\epsilon}{t_R} \left\{ 1 - \left[ a(\epsilon) - \frac{\left( \frac{m}{\kappa} \right)^\epsilon}{\epsilon} \right] t_R \right\}, \quad (29)
 \end{aligned}$$

up to one loop. In the minimal subtraction scheme, we use  $a(\epsilon)$  to subtract the pole only, i.e.,

$$a(\epsilon) - \frac{1}{\epsilon} \ln \left( \frac{m}{\kappa} \right)^\epsilon = 0 \quad (30)$$

Writing  $e^{\ln \left( \frac{m}{\kappa} \right)^\epsilon} \approx 1 + \epsilon \ln \frac{m}{\kappa}$ , we find

$$a(\epsilon) = 1/\epsilon. \quad (31)$$

Therefore, the redefinition of coupling constant  $T$  as

$$T = \frac{t_R}{\kappa^\epsilon} \left[ 1 + \frac{t_R}{\epsilon} \right], \quad (32)$$

will render the vertex functions finite. Up to and including one loop perturbation theory, we need only coupling constant renormalisation,

$$T = \kappa^{-\epsilon} \left[ t_R + \frac{t_R^2}{\epsilon} \right], \quad (33)$$

and mass renormalisation,

$$\frac{m^2}{T} = \frac{m_R^2 \kappa^\epsilon}{t_R} \quad (34)$$

In fact, the renormalisation of interface model in “ $1 + \epsilon$ ” dimensions has been reportedly done for up to two loops<sup>10</sup> and four loops” in certain cases. A different approach of perturbation theory<sup>7</sup> has also confirmed that the structure of renormalisation described here is valid up to arbitrary order of perturbation theory. Note that wave function renormalisation is not needed for the interface model. This is because of the fact that the dimension of the field “ $f$ ” is length, and Ward identities would ensure this up to all orders in perturbation theory”.

The Callan-Symanzik equation is given, for all vertex functions, as

$$\left[ \kappa \frac{\partial}{\partial \kappa} + \beta(t_R) \frac{\partial}{\partial t_R} + \gamma_1(t) m_R^2 \frac{\partial}{\partial m_R^2} \right] \Gamma_R = 0, \quad (35)$$

where the beta function and anomalous dimension are

$$\begin{aligned} \beta(t_R) &= \epsilon t_R - t_R^2 \\ \gamma_1(t_R) &= -\epsilon + \frac{\beta}{t_R}, \end{aligned} \quad (36)$$

up to one loop. There are two fixed points in this theory: one is an infrared stable (trivial) fixed point,  $t_R = 0$ , which controls the-mean field low momentum behaviour of the theory; the other is an ultraviolet stable fixed point,  $t_R = \epsilon$ , which is an effective coupling for the high momentum behaviour. This ultraviolet fixed point  $t_R$  is interpreted by WZ<sup>10</sup> as the critical temperature,  $T_c$ . They argue that the critical temperature  $T_c$  is given by,

$$T_c = \epsilon - \frac{\epsilon^2}{2} + O(\epsilon^3), \quad (37)$$

up to two loops. One of the physical motivations of WZ’s work is related to approaching

the critical point from below  $T_c$ , employing the formalism which uses the lower critical dimension of a system with discrete symmetry as the starting point (i.e. “ $1 + \epsilon$ ”-expansion). However, the formalism we develop here and elsewhere<sup>7</sup> proves to be useful to treat more general higher derivative geometric interactions. We shall now turn to the treatment of a small perturbation, due to “ $\sqrt{g} \cdot b$ ” insertion, to the interface model.

## II. SMALL PERTURBATIONS TO INTERFACE MODEL

The Hamiltonian, Equation (1), used in the interface model has a long history of use in surface and interface phenomenology<sup>15</sup>. In recent years, it has been shown that the interface Hamiltonian is derivable from a more fundamental Ginsburg-Landau-Wilson (GLW) Hamiltonian<sup>12,7,16</sup>. Since the field theoretical study of critical phenomena is based on this GLW Hamiltonian, it is reasonable to argue that the interface model is an effective Hamiltonian for a system displaying two-phase coexistence at the low temperature. The leading order term, in the long distance limit, is the surface area term<sup>12</sup> and the higher order contributions are also discovered in a systematic fashion<sup>7,16</sup>. They are given by the following effective Hamiltonian,

$$H_{\text{eff}} = \int d^{d-1} y \sqrt{1 + (\nabla f)^2} \left( \frac{1}{T} + c \mathbf{g} \cdot \mathbf{b} + dR \right) \quad (38)$$

where  $\mathbf{g} \cdot \mathbf{b} = [1 + (\nabla f)^2]^{-3/2} \partial_i f \partial_{ij} f \partial_j f$

$$\begin{aligned} R &= [1 + (\nabla f)^2]^{-1} (\partial_{ij} f \partial_{ij} f - \partial_{ii} f \partial_{jj} f) \\ \frac{1}{T} &= \int dz \left( \frac{d\phi_c(z)}{dz} \right)^2 \\ C &= \int dz z \left( \frac{d\phi_c(z)}{dz} \right)^2 \\ d &= \int dz z^2 \left( \frac{d\phi_c(z)}{dz} \right)^2 \end{aligned} \quad (39)$$

with  $(\mathbf{g} \cdot \mathbf{b})$  representing the contraction of the second fundamental form with the metric tensor  $g_{ij}$  and  $R$  representing the curvature scalar. The simplest perturbation to the interface model to be considered is, of course, the “ $\sqrt{g} \cdot b$ ” term

$$H_{\text{eff}}(f) = \int d^{d-1} y \sqrt{g} \left( \frac{1}{T} + c \mathbf{g} \cdot \mathbf{b} \right) + \frac{1}{2} m^2 f^2, \quad (40)$$

where the mass term is introduced as an infrared mass regulator,  $\sqrt{g} \equiv \sqrt{1 + (\nabla f)^2}$  and  $c$  is treated as a small number.

From the discussion of the previous section, we learn that the surface tension part (i.e. the surface area term) of the Hamiltonian is perturbatively renormalisable in an  $\epsilon$ -expansion context for  $\epsilon = d - 1$  with an essential coupling constant (T) renormalisation. A renormalised dimensionless coupling constant  $t_R$  is introduced,

$$T = Z \frac{t_R}{\kappa^\epsilon} \quad , \quad (41)$$

with the renormalisation constant,

$$Z = 1 + \frac{t_R}{\epsilon} + \frac{t_R^2}{\epsilon^2} \left(1 + \frac{\epsilon}{4}\right) + O(t_R^3) \quad , \quad (42)$$

up to two loops<sup>10</sup>. For the renormalisation of small perturbation,  $\sqrt{g} \mathbf{g} \cdot \mathbf{b}$ , we found a similar renormalisational structure to the surface tension part by employing an extended method of the effective potential<sup>7</sup>. We need an extra renormalisation for the coefficient  $c$  of “ $\sqrt{g} \mathbf{g} \cdot \mathbf{b}$ ”,

$$C^b = Z_c C^R \quad (43)$$

where  $C^b$  stands for “bare” and  $C^R$  for renormalised.  $Z_c$  is given, up to one loop, by

$$Z_c = 1 - \frac{t_R}{\epsilon} + O(t_R^2) \quad . \quad (44)$$

We will proceed to calculate the renormalisation of “ $\sqrt{g} \mathbf{g} \cdot \mathbf{b}$ ” by the conventional field theoretical method. This involves calculating any  $n$ -point vertex functions due to the small perturbations, and then employing the techniques of renormalisation and renormalisation group. We shall now present such a typical calculation of three-point vertex function. Techniques developed for the three-point function would apply to any  $n$ -point functions and they all have the same renormalisation procedure. This is guaranteed by the effective potential calculation<sup>7</sup>. For the three-point function up to one loop, we have four diagrams to calculate. The first one is

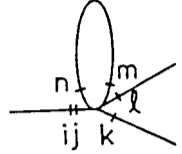
$$\begin{aligned} \text{Diagram} &= 2 \frac{1}{2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \partial_{ij} f \partial_k f \partial_l f \\ &= 2 \partial_{ij} f \partial_i f \partial_j f \quad , \end{aligned} \quad (45)$$

where the coordinate form “ $\partial_{ij} f \partial_i f \partial_j f$ ” is kept in order to emphasise the extra derivative of the small perturbation,  $\sqrt{g} \mathbf{g} \cdot \mathbf{b}$ . The expansion of “ $\sqrt{g} \mathbf{g} \cdot \mathbf{b}$ ” will give an infinite series of interacting vertices,

$$\sqrt{g} \mathbf{g} \cdot \mathbf{b} = [1 + (\nabla f)^2]^{-1} \partial_{ij} f \partial_i f \partial_j f$$

$$= [1 - (\nabla f)^2 + (\nabla f)^4 - (\nabla f)^6 + \dots] \partial_{ij} f \partial_i f \partial_j f \quad (46)$$

For our purpose of one loop calculation here, we only need to consider



$$= -12 F_{ijk\ell mn} \partial_{ij} f \partial_k f \partial_\ell f \partial_m f \partial_n f$$

$$\times \frac{T}{(2\pi)^{d-1}} \int \frac{k_m k_n d^{d-1} k}{k^2 + m^2} \quad (46)$$

where  $F_{ijk\ell mn}$  is not the sixth-order symmetric tensor,  $S_{ijk\ell mn}$ , but is given instead by

$$F_{ijk\ell mn} = \frac{1}{12} \left\{ (\delta_{ik} \delta_{jm} + \delta_{jk} \delta_{im}) \delta_{\ell n} + (\delta_{ik} \delta_{jn} + \delta_{jk} \delta_{in}) \delta_{\ell m} \right.$$

$$+ (\delta_{i\ell} \delta_{jm} + \delta_{j\ell} \delta_{im}) \delta_{kn} + (\delta_{i\ell} \delta_{jn} + \delta_{j\ell} \delta_{in}) \delta_{km}$$

$$\left. + (\delta_{ik} \delta_{j\ell} + \delta_{i\ell} \delta_{jk}) \delta_{mn} + (\delta_{im} \delta_{jn} \delta_{in} \delta_{jm}) \delta_{k\ell} \right\}$$

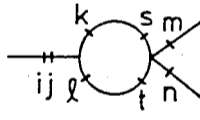
$$= \frac{1}{12} (15 S_{ijk\ell mn} - 3 \delta_{ij} S_{k\ell mn})$$

$$= \frac{1}{4} (5 S_{ijk\ell mn} - \delta_{ij} S_{k\ell mn}) \quad (47)$$

The contribution for this diagram is, therefore, written as

$$\frac{Tm^\epsilon}{\epsilon} [ 2 \partial_{ii} f \partial_k f \partial_k f + 8 \partial_{ij} f \partial_i f \partial_j f ] \quad (48)$$

We still have to consider the diagram contracted by the surface area term and the small perturbation term. We only have to calculate up to the first order perturbation theory for the one-loop case. There are two diagrams of this sort. The first one is,



$$= \frac{24}{2} (\delta_{ik} \delta_{j\ell} + \delta_{i\ell} \delta_{jk}) \partial_{ij} f$$

$$\times S_{stmn} \times \frac{1}{8T} \partial_m f \partial_n f$$

$$\times \frac{T^2}{T} \frac{1}{(2\pi)^{d-1}} \int \frac{k_k k_s k_t k_\ell d^{d-1} k}{(k^2 + m^2)[(k - q)^2 + m^2]}$$

$$= \frac{3T}{2} (\delta_{ik} \delta_{j\ell} + \delta_{i\ell} \delta_{jk}) S_{stmn} \partial_{ij} f \partial_m f \partial_n f$$

$$\times \frac{1}{(2\pi)^{d-1}} \int \frac{k_k k_s k_t k_\ell d^{d-1} k}{(k^2 + m^2)[(k - q)^2 + m^2]} \quad (49)$$

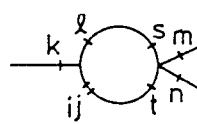
where the following trick will be used to simplify the diagram calculation,

$$\int d^d p p_i p_j p_k p_\ell f(p^2) = \frac{3 S_{ijkl}}{d(d+2)} \int d^d p (p^2)^2 f(p^2) \quad (50)$$

We obtain the contribution for this diagram as

$$-\frac{Tm^\epsilon}{\epsilon} [2 \partial_{ii} f \partial_m f \partial_m f + 2 \partial_{ij} f \partial_i f \partial_j f] \quad (51)$$

The other one for the first order perturbation theory to consider is



$$= \frac{48}{2} (\delta_{ik} \delta_{j\ell} + \delta_{i\ell} \delta_{jk}) \partial_k f$$

$$\times S_{stmn} \left(\frac{1}{8T}\right) \cdot \partial_m f \partial_n f$$

$$\times \int \frac{k_\ell k_i k_j k_s k_t d^{d-1} k}{(k^2 + m^2)[(k - q)^2 + m^2]} \quad (52)$$

the standard techniques of dimensionally regularised integrals, various identities, etc. will have to be used. The naive coordinate form of this diagram is “ $\partial_k f \partial_m f \partial_n f$ ”, which does not have an extra derivative. Therefore, this diagram is not supposed to contribute, naively. However, by actually calculating the Feynman integrals of this diagram, one would find that typical contribution might look like

$$2 q_k q_s q_t S_{stmn} \partial_k f \partial_m f \partial_n f$$

$$= \frac{3}{2} q^2 \partial_{kk} f \partial_m f \partial_m f + \frac{4}{3} q^2 \partial_{mn} f \partial_m f \partial_n f \quad (53)$$

After a long and tedious calculation, one would obtain the contribution for this diagram as

$$-\frac{Tm^\epsilon}{\epsilon} [4 \partial_{ij} f \partial_i f \partial_j f] \quad (54)$$

The total one-loop sum for the three-point function is, therefore, give by

$$2C \left(1 + \frac{Tm^\epsilon}{\epsilon}\right) \partial_{ij} f \partial_i f \partial_j f \quad (55)$$

Forms like this are similar to those which appeared in the interface model. This pattern of contribution repeats in any n-point vertex function, and only one extra renormalisation for

the small coupling constant,  $C$ , is needed. This confirms the effective potential calculations we have done before. The small perturbation,  $\sqrt{g} \mathbf{g} \cdot \mathbf{b}$ , to the surface tension Hamiltonian (interface model) is thus confirmed to be irrelevant under renormalisation group flows for small  $\epsilon$ .

## APPENDIX

We shall summarise, in this appendix, the formulae we have derived for symmetric tensor manipulation and other useful formulae.

$$S_{ijk\ell} = \frac{1}{3} (\delta_{ij} \delta_{k\ell} + \delta_{ik} \delta_{j\ell} + \delta_{i\ell} \delta_{jk}) \quad (\text{A1})$$

$$S_{ijk\ell mn} = \frac{1}{15} (\delta_{ij} \delta_{k\ell} \delta_{mn} + 14 \text{ terms}) \quad (\text{A2})$$

$$\underbrace{S_{ijk\ell \dots mn}}_{2n\text{-th order}} = \frac{1}{N} (\delta_{ij} \delta_{k\ell} \dots \delta_{mn} + (N-1) \text{ terms})$$

with

$$N = \frac{2n!}{n! 2^n} \quad (\text{A3})$$

The number of terms in a symmetric tensor,  $N$ , is derived by considering the problem, "How many ways can one put  $2n$  objects into  $n$  boxes, with two objects in each box?" . There is also a set of simplified forms for  $N$ ,

$$\begin{aligned} N &= 3, & n &= 2 \\ N &= 5 \cdot 3, & n &= 3 \\ N &= 7 \cdot 5 \cdot 3, & n &= 4 \\ N &= 9 \cdot 7 \cdot 5 \cdot 3, & n &= 5 \\ N &= 11 \cdot 9 \cdot 7 \cdot 5 \cdot 3, & n &= 6, \text{ etc,} \\ N &= (2n - 1)!! \text{ for any } n. & & \end{aligned} \quad (\text{A4})$$

The contractions of symmetric tensors are important formulae in field theory calculations, in general.

$$S_{ijkk} = \frac{d+2}{3} \delta_{ij} \quad (\text{A5})$$

$$S_{ijk\ell} S_{ijk\ell} = \frac{d(d+2)}{3} \quad (\text{A6})$$

$$S_{ijij} = \frac{d(d+2)}{3} \quad (\text{A7})$$

$$S_{ijmn} S_{mnk\ell} = \frac{1}{3} \left( \frac{(d+2)}{3} \delta_{ij} \delta_{k\ell} + 2 S_{ijk\ell} \right) \quad (\text{A8})$$

$$S_{ijk\ell mn} = \frac{d+4}{5} S_{ijk\ell} \quad (\text{A9})$$

$$S_{ijk\ell mn\infty} = \frac{d+6}{7} S_{ijk\ell mn} \quad (\text{A10})$$

The higher order and more complex contractions can all be derived accordingly. We will not list the standard dimensionally regularised integrals since they can be found in standard books<sup>14</sup> on modern quantum field theory. However, we will write down two more general formulae which are useful to those who would like to calculate the integrals instead of tabulating them from books.

$$\int d^d k k_\mu k_\nu f(k^2) = \frac{\delta_{\mu\nu}}{d} \int k^2 f(k^2) d^d k \quad (\text{A11})$$

$$\int d^d p p_i p_j p_k p_\ell f(p^2) = \frac{3 S_{ijk\ell}}{d(d+2)} \int d^d p (p^2)^2 f(p^2) \quad (\text{A12})$$

#### REFERENCES

1. S. C. Lin (1987), "Field Theories of Surfaces and Interfaces", in the Proceedings of 1986 Taipei Summer School on Statistical Physics, Phys. Society of R.O.C.
2. R. K. P. Zia (1984), "Interfacial Problems in Statistical Physics" in the Proceedings of SUSSP 1983, Dept. of Physics, Univ. of Edinburgh, U. K.
3. D. J. Wallace (1983), "Perturbative Approach to Surface Fluctuations", Les Houches Summer School 1982, North Holland.
4. D. J. Wallace (1985), "Non-perturbative Renormalisation in Field Theory", appeared in "Applications of Field Theory to Statistical Mechanics", Lecture Notes in Physics, 2 16, Springer-Verlag.
5. A. D. Bruce and D. J. Wallace (1981), Phys. Rev. Lett., Vol. 47, pp. 743, and J. Phys. A16, pp. 1721 (1983).
6. S. C. Lin (1984), Ph. D. Thesis, Univ. of Edinburgh. A more detailed account will be

published separately.

7. S. C. Lin and M. J. Lowe (1984), *Journal of Physics A: Math. Gen.*, Vol. 16, pp. 347,
8. D. A. Huse, W. van Saarloos and J. D. Weeks (1985), *Phys. Rev. B* 32, pp. 233.
9. R. K. P. Zia (1986), "Stability of the Isotropic Fixed Point near One Dimension", Virginia Polytechnic Institute and State Univ. Preprint, Submitted to *Nuclear Phys. B*.
10. D. J. Wallace and R. K. P. Zia (1979), *Phys. Rev. Lett.*, Vol. 43, pp. 808.
11. D. Forster and A. Gabriunas (1981), *Phys. Rev. A* 24, pp. 598.
12. N. J. Gunther, D. A. Nicole, D. J. Wallace (1980), *J. Phys. A* 13, pp. 1755.
13. D. J. Amit (1984), "Field Theory, the Renormalisation Group, and Critical Phenomena", Second Edition, World Scientific.
14. C. Itzykson and J-B Zuber (1980), "Quantum Field Theory", McGraw-Hill.
15. F. P. Buff, R. A. Lovett, F. H. Stillinger (1965), *Phys. Rev. Lett.*, Vol. 15, pp. 621.
16. R. K. P. Zia (1985), *Nuclear Physics B* 251, pp. 676.