Necessary and Sufficient Conditions for the Existence of Metric in Two-Dimensional Affine Manifolds*

KUO-SHUNG CHENG (鄭國順)

Department of Applied Mathematics, National Chico Tang University
Hsinchu, Taiwan

and

WEI-TOU NI (倪維斗)

Department of Physics, National Tsing Hua University
Hsinchu, Taiwan

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Necessary and sufficient conditions for the existence of metric in two-dimensional affine manifolds are found to be (i) $R_{12}=R_{21}$ (ii) $R^1_{112}R^2_{112;1}=R^2_{112}R^1_{112;1}$ (iii) $R^1_{112}R^2_{112;2}=R^2_{112}R^1_{112;2}$ (iv) $R^1_{112}R^1_{212;1}=R^1_{212}R^1_{112;1}$ (v) $R^1_{112}R^1_{212;2}=R^1_{212}R^1_{112;2}$, where $R^{\alpha}_{\beta\gamma\delta}$ and $R_{\beta\delta}$ are respectively the Riemann tensor and the Ricci tensor of the manifold. In case the above five conditions are satisfied, the solutions for metric are found.

I. INTRODUCTION

DIFFERENTIAL geometry is playing an increasingly important role in theoretical physics. In some applications one starts with an affine connection and obtains an affine manifold. While in some other applications one starts with a metric and obtains a (pseudo-)Riemannian manifold. Every Riemannian manifold is also an affine manifold, but the converse is not true. Riemannian manifolds have richer properties than affine manifold. Therefore it is desirable to find the necessary and sufficient conditions for affine manifolds to have a metric. This is also of intrinsic mathematical importance.

Gravitation and geometry have a very close relationship. Galileo's weak equivalence principle (Universality of Free Fall) tells us that there is a preferred family of curves called free-fall trajectoies. With some general requirements on the properties of these curves, one can regard them as geodesics in the 4-dimensional spactime. With these geodesics, one defines an affine manifold. Newtonian gravitation in Cartan's form[1] defines such an affine manifold which can not be metrized. If one postulates Einstein equivalence principle and the validity of special relativity "in the absence of gravity", then one can go further to obtain a metric and the spacetime becomes Riemannian. Universality of Free Fall is verified to a high precision while Einstein equivalence principle is only verified in some aspects to about $10^{-2}-10^{-3}$ accuracy. Furthermore, according to a recent analysis[2], Universality of Free Fall plus special relativity does not imply Einstein equivalence principle. Therefore it is desirable to investigate what are the mathematical conditions for an affine manifold to have a metric.

Electromagnetism and general relativity are gauge-type theories. Up to now the long standing efforts to unify them seem to be fruitless. But the recent gauge-theoretic unification of electromagne-

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(1) É. Cartan, Ann. École Norm. Sup. *40*, 32(1923); *41*, 1 *(1924). See* also, Chapter *12* of C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973) for an excellent treatment.

(2) W.-T. Ni, Bull. Am. Phys. Soc. 19, 655 (1974); Phys. Rev. Lett. 38, 301 (1977).

tism and weak interaction seems to be quite fruitful. In view of this, efforts are made to bring the gravitation into the present gauge-theoretic framework[3]. Along this line, the affine connections correspond to gauge potentials in other gauge theories and should be the dynamical variables in the Lagrangian formulation. With this approach, it is only natural that the metric be not postulated but arrived at from field equations or some other natural considerations. Hence necessary and sufficient conditions for the metric to exist are helpful in this investigation.

To start with, we obtain the necessary and sufficient conditions for the existence of metric in *two-dimensional* affine manifolds in this paper.

II. NECESSARY AND SUFFICIENT CONDITIONS

Given a two-dimensional affine manifold $\mathfrak{M}$ with (symmetric) affine connection $\Gamma^{\alpha}_{\beta\gamma}(x, y)$ and Riemannian tensor

$$R^{\alpha}_{\beta\gamma\delta}(x, y)\equiv\Gamma^{\alpha}_{\beta\delta,\gamma}-\Gamma^{\alpha}_{\beta\gamma,\delta}+\Gamma^{\alpha}_{\mu\gamma}\Gamma^{\mu}_{\beta\delta}-\Gamma^{\alpha}_{\mu\delta}\Gamma^{\mu}_{\beta\gamma}\,, \tag{1}$$

we will, in this section, derive the necessary and sufficient conditions for a (nonsingular) metric $g_{\alpha\beta}$ to exist such that the affine connection can be obtained from this metric as in a Riemannian manifold, i. e.,

$$\Gamma^{\alpha}_{\beta\gamma}=\frac{1}{2}g^{\alpha\delta}(g_{\beta\delta,\gamma}+g_{\delta\gamma,\beta}-g_{\beta\gamma,\delta}). \tag{2}$$

In the following, we use the convention that "," denotes partial differentiation and ";" denotes covariant differentiation.

Equations (2) are equivalent to

$$g_{\alpha\beta,\gamma}=\Gamma^{\mu}_{\alpha\gamma}g_{\mu\beta}+\Gamma^{\mu}_{\beta\gamma}g_{\alpha\mu} \tag{3}$$

which in component form are

$$g_{11,1}=2\Gamma^{1}_{11}g_{11}+2\Gamma^{2}_{21}g_{12}\,, \tag{4}$$

$$g_{11,2}=2\Gamma^{1}_{12}g_{11}+2\Gamma^{2}_{12}g_{12}\,, \tag{5}$$

$$g_{22,1}=2\Gamma^{1}_{12}g_{12}+2\Gamma^{2}_{12}g_{22}\,, \tag{6}$$

$$g_{22,2}=2\Gamma^{1}_{22}g_{12}+2\Gamma^{2}_{22}g_{22}\,, \tag{7}$$

$$g_{12,1}=\Gamma^{1}_{12}g_{11}+(\Gamma^{1}_{11}+\Gamma^{2}_{12})g_{12}+\Gamma^{2}_{11}g_{22}\,, \tag{8}$$

$$g_{12,2}=\Gamma^{1}_{22}g_{11}+(\Gamma^{1}_{12}+\Gamma^{2}_{22})g_{12}+\Gamma^{2}_{12}g_{22}\,. \tag{9}$$

Equations (4)–(9) are six partial differential equations in three unknowns g_{11}, g_{12}, g_{22}. We will derive the integrability conditions for them in the following.

Differentiating (3) with respect to δ and substituting (3) back for the first derivatives of $g_{\alpha\beta}$, we obtain

$$\begin{aligned} g_{\alpha\beta,\gamma\delta}&=g_{\mu\beta}(\Gamma^{\mu}_{\alpha\gamma,\delta}+\Gamma^{\nu}_{\alpha\gamma}\Gamma^{\mu}_{\nu\delta}) \\ &\quad+g_{\alpha\mu}(\Gamma^{\mu}_{\beta\gamma,\delta}+\Gamma^{\nu}_{\beta\gamma}\Gamma^{\mu}_{\nu\delta}) \\ &\qquad+g_{\mu\nu}(\Gamma^{\mu}_{\alpha\gamma}\Gamma^{\nu}_{\beta\delta}+\Gamma^{\mu}_{\beta\gamma}\Gamma^{\nu}_{\alpha\delta}). \end{aligned} \tag{10}$$

From $g_{\alpha\beta,\gamma\delta}=g_{\alpha\beta,\delta\gamma}$ and (10),

$$g_{\mu\beta}R^{\mu}_{\alpha\delta\gamma}+g_{\alpha\mu}R^{\mu}_{\beta\delta\gamma}=0\,. \tag{11}$$

In component form, equation (11) is

(3) See, e.g.. C. N. Yang, Phys. Rev. Lett. 33, 445 (1974).

$$R^1_{112}g_{11}+R^2_{112}g_{12}=0\,, \tag{12}$$

$$R^1_{212}g_{11}+(R^1_{112}+R^2_{212})g_{12}+R^2_{112}g_{22}=0\,, \tag{13}$$

$$R^1_{212}g_{12}+R^2_{212}g_{22}=0\,. \tag{14}$$

Note that (11) involve the unknows $g_{\alpha\beta}$, so they are equations, not integrability conditions"). (12)-(14) are three linear homogeneous equations in three unknowns; in order to have a nontrivial solution, we must have

$$\det\begin{pmatrix} R^1_{112} & R^2_{112} & 0 \\ R^1_{212} & (R^1_{112}+R^2_{212}) & R^2_{112} \\ 0 & R^1_{212} & R^2_{212}\end{pmatrix} = (R^1_{112}+R^2_{212})(R^2_{112}R^2_{212}-R^1_{112}R^1_{212}) = 0$$

Since in general

$$R^1_{112}R^2_{212}-R^2_{112}R^1_{212}\neq 0 \tag{16}$$

we must have

$$R^1_{112}+R^2_{212}=0\,. \tag{17}$$

(17) is an integrability condition. It is equivalent to

$$R_{12}=R_{21}\quad\text{or}\quad R_{\alpha\beta}=R_{\beta\alpha} \tag{17'}$$

where

$$R_{\alpha\beta}\equiv R^{\gamma}_{\alpha\gamma\beta}\,. \tag{18}$$

(17') can also be derived in the usual fashion as given in differential geometry textbooks. If (17) is satisfied, (12)-(14) can be solved for the ratios of g's:

$$g_{12}=-\frac{R^1_{112}}{R^2_{112}}g_{11}=-\frac{R^2_{212}}{R^1_{212}}g_{22}\,. \tag{19}$$

Substituting (19) into (4)-(9), we obtain

$$\frac{\partial \ln|g_{11}|}{\partial x}=2\left(\Gamma^1_{11}-\frac{R^1_{112}}{R^2_{112}}\Gamma^2_{11}\right), \tag{20}$$

$$\frac{\partial \ln|g_{11}|}{\partial y}=2\left(\Gamma^1_{12}-\frac{R^1_{112}}{R^2_{112}}\Gamma^2_{12}\right), \tag{21}$$

$$\frac{\partial \ln|g_{22}|}{\partial x}=2\left(\Gamma^2_{12}-\frac{R^2_{212}}{R^1_{212}}\Gamma^1_{12}\right), \tag{22}$$

$$\frac{\partial \ln|g_{22}|}{\partial y}=2\left(\Gamma^2_{22}-\frac{R^2_{212}}{R^1_{212}}\Gamma^1_{22}\right), \tag{23}$$

$$\frac{\partial \ln|g_{12}|}{\partial x}=\left(\Gamma^1_{11}+\Gamma^2_{12}-\frac{R^2_{112}}{R^1_{112}}\Gamma^1_{12}-\frac{R^1_{212}}{R^2_{212}}\Gamma^2_{11}\right), \tag{24}$$

$$\frac{\partial \ln|g_{12}|}{\partial y}=\left(\Gamma^1_{12}+\Gamma^2_{22}-\frac{R^1_{112}}{R^2_{112}}\Gamma^1_{22}-\frac{R^1_{212}}{R^2_{212}}\Gamma^2_{12}\right). \tag{25}$$

The necessary and sufficient conditions for (20)-(23) to be integrable are

$$\frac{\partial}{\partial y}\left(\Gamma^1_{11}-\frac{R^1_{112}}{R^2_{112}}\Gamma^2_{11}\right)-\frac{\partial}{\partial x}\left(\Gamma^1_{12}-\frac{R^1_{112}}{R^2_{112}}\Gamma^2_{12}\right)=0\,, \tag{26}$$

$$\frac{\partial}{\partial y}\left(\Gamma^2_{12}-\frac{R^2_{212}}{R^1_{212}}\Gamma^1_{12}\right)-\frac{\partial}{\partial x}\left(\Gamma^2_{22}-\frac{R^2_{212}}{R^1_{212}}\Gamma^1_{22}\right)=0\,. \tag{27}$$

(4) C. M. Pereira, J. Math. Phys. 13, 1542 (1972).

(26) and (27) are equivalent to (we use (17) freely.)

$$\Gamma^2_{11}[R^1_{112}R^2_{112;2}-R^2_{112}R^1_{112;2}]+\Gamma^2_{12}[R^2_{112}R^1_{112;1}-R^1_{112}R^2_{112;1}]=0\,, \tag{28}$$

$$\Gamma^1_{12}[R^1_{212}R^1_{112;2}-R^1_{112}R^1_{212;2}]+\Gamma^1_{22}[R^1_{112}R^1_{212;1}-R^1_{212}R^1_{112;1}]=0\,. \tag{29}$$

(28) and (29) are necessary conditions for metric to exist. Choose a coordinate system such that $\Gamma^2_{12}=0$, but $\Gamma^2_{11}\neq 0$, at point P. Then at this point (28) becomes

$$R^1_{112}R^2_{112;2}-R^2_{112}R^1_{112;2}=0\,. \tag{30}$$

Since point P could be arbitrary, (30) must holds at every point. Similarly,

$$R^2_{112}R^1_{112;1}-R^1_{112}R^2_{112;1}=0\,, \tag{31}$$

$$R^1_{212}R^1_{112;2}-R^1_{112}R^1_{212;2}=0\,, \tag{32}$$

$$R^1_{112}R^1_{212;1}-R^1_{212}R^1_{112;1}=0\,, \tag{33}$$

are necessary conditions. If (17), (30)–(33) hold, then (28), (29) hold and (20)–(23) can be integrated. Moreover, if (17), (30)–(33) hold, then we can demonstrate that (24) and (25) can be integrated. That is, (24) and (25) give no additional integrability condition.

Therefore we have proved the following theorem:

Theorem : The necessary and sufficient conditions for the existence of metric in two-dimensional affine manifolds are

(i) $R_{12}=R_{21}$,

(ii) $R^1_{112}R^2_{112;1}=R^2_{112}R^1_{112;1}$,

(iii) $R^1_{112}R^2_{112;2}=R^2_{112}R^1_{112;2}$,

(iv) $R^1_{112}R^1_{212;1}=R^1_{212}R^1_{112;1}$,

(v) $R^1_{112}R^1_{212;2}=R^1_{212}R^1_{112;2}$.

In case the above conditions are satisfied the solutions are

$$g_{11}=\exp\left\{2\int_{(x_0,y_0)}^{(x,y)}\left(\Gamma^1_{11}-\frac{R^1_{112}}{R^2_{112}}\Gamma^2_{11}\right)dx+\left(\Gamma^1_{12}-\frac{R^1_{112}}{R^2_{112}}\Gamma^2_{12}\right)dy+C_1\right\}, \tag{34}$$

$$g_{22}=\exp\left\{2\int_{(x_0,y_0)}^{(x,y)}\left(\Gamma^2_{12}-\frac{R^2_{112}}{R^1_{212}}\Gamma^1_{12}\right)dx+\left(\Gamma^2_{22}-\frac{R^2_{112}}{R^1_{212}}\Gamma^1_{22}\right)dy+C_2\right\}, \tag{35}$$

$$g_{12}=\exp\left\{\int_{(x_0,y_0)}^{(x,y)}\left(\Gamma^1_{11}+\Gamma^2_{12}-\frac{R^2_{112}}{R^1_{112}}\Gamma^1_{12}-\frac{R^1_{212}}{R^1_{112}}\Gamma^2_{11}\right)dx + \left(\Gamma^1_{12}+\Gamma^2_{22}-\frac{R^2_{112}}{R^1_{112}}\Gamma^1_{22}-\frac{R^1_{212}}{R^2_{112}}\Gamma^2_{12}\right)dy+C_3\right\}, \tag{36}$$

where C_1, C_2 and C_3 must be properly selected in order to satisfy (12), (13) and (14). We can prove that the constants C_1 and C_2 depend on C_3 and that C_3 is arbitrary. This arbitrary constant C_3 corresponds to the constant scale change of the metric. This is consistent with the fact that the affine connetion Γ^k_{ij} are invariant under such a scale change.

III. DISCUSSION

(1) The above theorem can also be proved as follows: We regard (12), (14), (4) and (5) as equations for determining g_{11}, g_{12} and g_{22}. Thus we obtain integrability conditions (30) and (31). (32) and (33) can be obtained in a similar way. So long as (30) and (31) are satisfied, we can solve (4) and (5) to obtain g_{11} as in (34). Now we can use (12) and (14) to obtain g_{12} and g_{22} in terms of g_{11}. In order to satisfy (13), we must have (17) to be satisfied. We can also prove that the above solutions of g_{11}, g_{12} and g_{13} satisfy (6), (7), (8) and (9) provided that (17) and (30)–(33) are satisfied. Hence the sufficiency and the theorem is proved.

(2) The generalization to higher dimensional manifolds is not so obvious and straightforward. It may involve some complications. The three and four dimensional cases are currently under investigation.

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