Linear Theory of the $E\times B$ Instability

C. I. WENG(翁政義) and C. S. MA(馬中興)

Department of Mechanical Engineering, National Cheng Kung University, Tainan, Taiwan

(Received 18 February 1975)

A linear analysis of the $E\times B$ instability in cylindrical coordinates in a weakly ionized plasma is studied. It is shown that the qualitative feature of instability is in agreement with Simon's result of slab model. The system is unstable when the sign of the product of the electric field and the density gradient is positive and the applied potential is large enough. It is also shown that the oscillation frequencies of the excited modes fit Saito's experimental results very well in the quantitative feature.

I. INTRODUCTION

THERE are a number of low-frequency electrostatic instabilities existed in a weakly ionized plasma. Among these is one called the "$E\times B$ instability", or "crossed field instability". This instability, first derived by Simon[1], and Hoh[2] independently, occurs in the presence of a density gradient and a parallel electric field, both perpendicular to a magnetic field. The mechanism of the instability is due to different drift velocities of the ions and electrons in crossed electric and magnetic fields when finite resistivity is included. The theoretical calculation of Siomn[1] shows that if the product of the applied electric field E_0 times the density gradient is positive, and if E_0 exceeds a critical value, collisional damping is overcome and a flute-like instability occurs.

Experimental demonstration of this instability was treated by Saito *et al.*[3]. The experiments are carried out using a hot cathode discharge tube with a coaxial electrode confrigu.ration in a weak magnetic field perpendicular to a radial electric field. The plasma used is a dark plasma of the anode glow mode. As the magnetic field is increased and exceeds a certain critical value, the excitation of low-frequency oscillations propagating in the direction of $E_0\times B$ is found. The theoretical results of Simon[1] and Saito *et al.*[3] might explain the experimental data. However, as the theoretical results are derived using the slab geometry, some approximations for the numerical substitution have to be made; And since the slab model is not quite in agreement with the anode glow discharge tube, the theoretical results can not fit the experimental data well,. Therefore, a calculation for the realistic cylindrical geometry is necessary.

In this paper, we accept Simon's theoretical arguments[1] in general except

(1) A. Simon, Phys Fluids 6, 382 (1963).
(2) F. C. Hoh, Phys. Fluids 6. 1184 (1963).
(3) S. Saito, N. Sato, Y. Hatta, J. Phys. Soc. of Japan **21,** 2655 (1966).

for the alteration of slab geometry to cylindrical geometry with coaxial electrode boundaries. However, we shall see below that this alteration will present some difficulties in finding the equilibrium density distribution, i.e., the equilibrium solution of cylindrical model is not so simple as in slab model. To find the undetermined constants for steady state plasma density distribution, a comparison with the realistic experimental condition for the boundary conditions has to be made. Our final results, can check Simon's results[3] in qualitative feature. and mainly emphasize on the improvement of the quantitative feature compared with the experimental results.

II. BASIC EQUATIONS AND EQUILIBRIUM SOLUTION

Consider a weakly ionized plasma with no recombination or ionization in the plasma region itself in a magnetic field in axial direction, the basic equations are continuity equation and equation of motion for each species[4]:

$$\frac{\partial n^{\pm}}{\partial t}+\nabla\cdot\boldsymbol{\Gamma}^{\pm}=0, \tag{1}$$

$$\frac{\partial}{\partial t}(m^{\pm}\boldsymbol{\Gamma}^{\pm})+\nabla(n^{\pm}KT^{\pm})\mp q^{\pm}(n^{\pm}\boldsymbol{E}+\boldsymbol{\Gamma}^{\pm}\times\boldsymbol{B})=-\nu^{\pm}m^{\pm}\boldsymbol{\Gamma}^{\pm}, \tag{2}$$

where the superscripts $\pm$ represent for ions and electrons respectively. Eqs. (2), ignoring the inertia, lead at once to the usual transport equations as follows:

$$\begin{aligned}
\Gamma_r^{\pm}&=-D_{\perp}^{\pm}\frac{\partial n^{\pm}}{\partial r}\pm\mu_{\perp}^{\pm}n^{\pm}E_r\mp(\Omega\tau)_{\pm}\left(D_{\perp}^{\pm}\frac{1}{r}\frac{\partial n^{\pm}}{\partial\theta}\mp\mu_{\perp}^{\pm}n^{\pm}E_\theta\right),\\
\Gamma_\theta^{\pm}&=-D_{\perp}^{\pm}\frac{1}{r}\frac{\partial n^{\pm}}{\partial\theta}\pm\mu_{\perp}^{\pm}n^{\pm}E_\theta\mp(\Omega\tau)_{\pm}\left(-D_{\perp}^{\pm}\frac{\partial n^{\pm}}{\partial r}\pm\mu_{\perp}^{\pm}n^{\pm}E_r\right),\\
\Gamma_z^{\pm}&=-D^{\pm}\frac{\partial n}{\partial z}\pm\mu^{\pm}nE_z,
\end{aligned} \tag{3}$$

where n is the number density, Γ the averaged flow density, $\boldsymbol{E}$ the electric field. The positive quantities D and μ represent the usual diffusion and mobility coefficients, where $D_{\perp}$ and $\mu_{\perp}$ are the same quantities devided by $[1+(\Omega\tau)^2]$, where Ω is the cyclotron frequency and τ is the mean time for collision with the background gas. We make the usual assumption of quasineutrality $(n^+\cong n^-\cong n)$ as well as electrostatic nature of the electric field. The complete equation of motion for each species is obtained by substitution of Eqs. (3) in the Eq. (1). We find

$$\begin{aligned}
&\frac{\partial n}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}\left[r\left\{-D_{\perp}^{\pm}\frac{\partial n}{\partial r}\pm\mu_{\perp}^{\pm}nE_r\mp(\Omega\tau)_{\pm}\left(D_{\perp}^{\pm}\frac{1}{r}\frac{\partial n}{\partial\theta}\mp\mu_{\perp}^{\pm}nE_\theta\right)\right\}\right]\\
&+\frac{1}{r}\frac{\partial}{\partial\theta}\left[-D_{\perp}^{\pm}\frac{1}{r}\frac{\partial n}{\partial\theta}\pm\mu_{\perp}^{\pm}nE_\theta\mp(\Omega\tau)_{\pm}\left(-D_{\perp}^{\pm}\frac{\partial n}{\partial r}\pm\mu_{\perp}^{\pm}nE_r\right)\right]\\
&+\frac{\partial}{\partial z}\left[-D^{\pm}\frac{\partial n}{\partial z}\pm\mu^{\pm}nE_z\right]=0.
\end{aligned} \tag{4}$$

(4) H. Haskell, "*Plasma Dynamics*"; (The Macmillan Company, 1965), P. 244, 248-252, 265.

The equilibrium state is specified as having a density variation $n_0(r)$ which varies only as r and an electric field $E_0(r)$ in the r-direction. Eq. (4) then reduces to

$$-D_{\perp}^{\pm}\frac{d^2n_0}{dr^2}\pm\mu_{\perp}^{\pm}\frac{d(n_0E_0)}{dr}-\frac{1}{r}D_{\perp}^{\pm}\frac{dn_0}{dr}\pm\mu_{\perp}^{\pm}\frac{1}{r}n_0E_0=0. \tag{5}$$

Note the Eq. (5) represents two equations actually. By eliminating the second and fouth terms between Eqs. (5), it yields

$$\frac{d^2n_0}{dr^2}+\frac{1}{r}\frac{dn_0}{dr}=0.$$

The general solution of this equation is

$$n_0=C_1\ln r+C_2, \tag{6}$$

where C_1 and C_2 are constants to be determined by the boundary conditions. We utilize the model used by Johnson and Webster[(5)] in determining the plasma density distribution in the anode glow mode. The geometry and coordinate system is shown in Fig. 1.

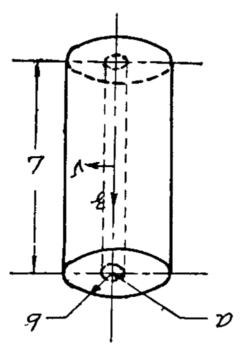

Fig. 1. Geometry and coordinate system used for the analysis of plasma density distribution.

We have assumed that the density distribution is function of r only, i. e., assumed a large value of L compared to that of b. This is equivalent to making the diffusion losses of ions to the end plates negligible. An immediate consequence of the first equation of Eqs. (3) is

$$\Gamma_r^{+}+\frac{\mu_{\perp}^{+}}{\mu_{\perp}^{-}}\Gamma_r^{-}=-D_a\frac{dn_0}{dr}.$$

For $\mu_{\perp}^{+}\ll\mu_{\perp}^{-}$, it is obvious that Γ_r^{+} is negligible at any radius r. If Γ_{eb} is the electron flow density at the anode surface $(r=b)$, and is introduced by the relation

$$\Gamma_r^{-}=\frac{b\Gamma_{eb}}{r}. \tag{7}$$

Then Es. (7) yields

(5) E. 0. Johnson, W.M.Webster, RCA Reviews, **82**, (1955)

$$C_1=-\frac{b\Gamma_{eb}}{D_a}\frac{\mu_\perp^+}{\mu_\perp^-},$$

where D_a is the usual ambipolar diffusion coefficient. As to the second boumdary condition, if n_b is the plasma density at the edge of the anode sheath, and is defined by the kinetic theory relation

$$\frac{1}{4}n_b\widetilde{C}_e=\Gamma_{eb},$$

then leads to

$$C_2=n_b+\frac{b\Gamma_{eb}\mu_\perp^+}{\mu_\perp^-}\text{Inb}. \tag{8}$$

Here, $\widetilde{C}_e$ is the mean thermal velocity of the plasma electron. Thus the steady state plasma density distribution is

$$n_0(r)=n_b\left[1+\frac{\widetilde{C}_e}{4}\frac{b}{D_a}\frac{\mu_\perp^+}{\mu_\perp^-}\ln\frac{b}{r}\right].$$

On the other hand, by elimination of the first and third terms between Eqs. (5), we find

$$\frac{d(n_0E_0)}{dr}+\frac{1}{r}n_0E_0=0.$$

Hence $E_0=C_3/n_0r$, where C_3 is a constant which can be determined by the requirement that

$$\int_a^b E_0dr=-V_0,$$

where V_0 is the applied electric potential. This leads at once to the result

$$E_0(r)=\frac{-V_0C_1}{r\ln\ [n_0(b)/n_0(a)]}\frac{1}{n_0(r)}. \tag{9}$$

The steady state is thus specified by Eqs. (8) and (9).

III. LINEAR ANALYSIS

We now make small perturbation about the steady state solution described above, i. e., let

$$n=n_0+n_1,$$
$$E=E_0-\nabla\varphi_1.$$

Here. n_1 and φ_1 are the perturbed density and electrostatic potential respectively. Then the linearized form of Eq. (4) becomes

$$\begin{aligned}&\frac{\partial n_1}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(-D_\perp^\pm\frac{\partial n_1}{\partial r}\pm\mu_\perp^\pm n_1E_0\mp\mu_\perp^\pm n_0\frac{\partial\varphi_1}{\partial r}\right)\right]\\&+\frac{1}{r}\frac{\partial}{\partial\theta}\left[-D_\perp^\pm\frac{1}{r}\frac{\partial n_1}{\partial\theta}\mp\mu_\perp^\pm n_0\frac{1}{r}\frac{\partial\varphi_1}{\partial\theta}\right]+\frac{\partial}{\partial z}\left[-D^\pm\frac{\partial n_1}{\partial z}\mp\mu^\pm n_0\frac{\partial\varphi_1}{\partial z}\right]\\&\pm(\Omega\tau)_\pm\left[\mp\mu_\perp^\pm\frac{dn_0}{dr}\frac{1}{r}\frac{\partial\varphi_1}{\partial\theta}\mp\mu_\perp^\pm E_0\frac{1}{r}\frac{\partial n_1}{\partial\theta}\right]=0.\end{aligned} \tag{10}$$

Since the steady state is function of r only, we can apply normal mode analysis at once to the perturbed quantities, i.e., we can Fourier analyze with respect to θ, z, and t. If the perturbation is of form

$$n_1 = n_1(r) \exp(-i\omega t + im\theta + ikz),$$
$$\varphi_1 = \varphi_1(r) \exp(-i\omega t + im\theta + ikz),$$

Then Es. (10) becomes

$$\left[-i\omega + D_\perp^\pm \frac{m^2}{r^2} + D^\pm k^2 - i(\Omega\tau)_\pm \mu_\perp^\pm \frac{m}{r} E_0 \pm \mu_\perp^\pm \frac{1}{r} E_0 \pm \mu_\perp^\pm \frac{dE_0}{dr}\right] n_1$$
$$+\left[\pm \mu_\perp^\pm E_0 - D_\perp^\pm \frac{1}{r}\right] \frac{dn_1}{dr} - D_\perp^\pm \frac{d^2 n_1}{dr^2} + \left[\pm n_0\left(\mu_\perp^\pm \frac{m^2}{r^2} + \mu^\pm k^2\right) -\right.$$
$$\left. i(\Omega\tau)_\pm \mu_\perp^\pm \frac{m}{r} \frac{dn_0}{dr}\right] \varphi_1 \mp \mu_\perp^\pm \left(\frac{n_0}{r} + \frac{dn_0}{dr}\right) \frac{d\varphi_1}{dr} \mp \mu_\perp^\pm n_0 \frac{d^2\varphi_1}{dr^2} = 0. \tag{11}$$

The equations above constitute two sets of coupled differential equations in the unknowns $n_1(r)$ and $\varphi_1(r)$. They are to be solved subject to the boundary conditions

$$\varphi_1(a) = \varphi_1(b) = 0$$

and (12)

$$n_1(a) = n_1(b) = 0.$$

Eqs. (11) have nonconstant coefficients and are thus rather difficult to solve exactly. Instead, we shall use the method of Galerkin[6], and take sin $\pi x/d$, where $x = r - a$ and $d = b - a$, as the first term in an expansion of n_1 and φ_1 in sine approximation to the correct eigenfunction. Therefore, the simple trial function for n_1 and φ_1 which satisfy the boundary conditions are of the form

$$n_1(r) = \bar{n}_1 \sin \pi x/d,$$
$$\varphi_1(r) = \bar{\varphi}_1 \sin \pi x/d, \tag{13}$$

where the coefficients on the right-hand side are complex constants. We then substitute Eqs. (13) in Eqs. (11), multiply from the left by sin m/d and integrate over the radial coordinate. The result is set of a coupled algebraic equations for $\bar{n}_1$ and $\bar{\varphi}_1$. The resulting algebraic equations are

$$\left[-i\omega + 2D_\perp^\pm m^2 J + D^\pm k^2 - i(\Omega\tau)_\pm \mu_\perp^\pm m(\widetilde{E}_0)_1 \pm \mu(\widetilde{E}_0)_1\right.$$
$$\left.\pm \mu_\perp^\pm \left(\frac{d\widetilde{E}_0}{dr}\right) \pm \mu_\perp^\pm (\widetilde{\widetilde{E}}_0) - D_\perp^\pm J + D_\perp^\pm \left(\frac{\pi}{d}\right)^2\right] \widetilde{n}_1 +$$
$$\left[\pm \mu_\perp^\pm m^2 (\widetilde{n}_0)_2 \pm \mu^\pm k^2 (\widetilde{n}_0) - i(\Omega\tau)_\pm \mu_\perp^\pm m\left(\frac{d\widetilde{n}_0}{dr}\right)_1\right.$$
$$\left.\mp \mu_\perp^\pm (\widetilde{\widetilde{n}}_0)_1 \mp \mu_\perp^\pm \left(\frac{d\widetilde{\widetilde{n}}_0}{dr}\right) \pm \mu_\perp^\pm \left(\frac{\pi}{d}\right)^2 (\widetilde{n}_0)\right] \bar{\varphi}_1 = 0, \tag{14}$$

where

(6) L. V. Kantorovich and V.I. Krylov, "*Approximate* **Methods** *of* **Higher Analysis**"; (Intersciences Publishers, Inc., New York, 1958), p. 258.

$$J=\frac{2}{d}\int_0^d \frac{1}{r}\frac{d}{dx}\left(\sin\frac{\pi x}{d}\right)\sin\frac{\pi x}{d}\,dx,$$

$$(\tilde{f})=\frac{2}{d}\int_0^d f\sin^2\frac{\pi x}{d}\,dx,$$

$$(\tilde{f})_1=\frac{2}{d}\int_0^d \frac{1}{r}f\sin^2\frac{\pi x}{d}\,dx,$$

$$(\tilde{f})_2=\frac{2}{d}\int_0^d \frac{1}{r^2}f\sin^2\frac{\pi x}{d}\,dx,$$

$$(\tilde{\tilde{f}})=\frac{2}{d}\int_0^d f\frac{d}{dx}\left(\sin\frac{\pi x}{d}\right)\sin\frac{\pi x}{d}\,dx,$$

$$(\tilde{\tilde{f}})_1=\frac{2}{d}\int_0^d \frac{1}{r}f\frac{d}{dx}\left(\sin\frac{\pi x}{d}\right)\sin\frac{\pi x}{d}\,dx.$$

Denote

$$X_\pm=D_\perp^\pm\left[(2m^2-1)J+\left(\frac{\pi}{d}\right)^2\right]+D^\pm k^2,$$

$$Y_\pm=\mu_\perp^\pm\left(\frac{\pi}{d}\right)^2+\mu^\pm k^2,$$

$$F=(\tilde{E}_0)_1+\left(\frac{d\tilde{E}_0}{dr}\right)+(\tilde{\tilde{E}}_0),$$

$$G=m^2(\tilde{n}_0)_2-\left(\frac{d\tilde{\tilde{n}}_0}{dr}\right)-(\tilde{\tilde{n}}_0)_1.$$

Hence Eqs. (14) become

$$\begin{aligned}&[-i\omega+X_\pm\pm\mu_\perp^\pm F-i(\Omega\tau)_\pm\mu_\perp^\pm m(\tilde{E}_0)_1]\bar{n}_1+\\&\left[\pm(\tilde{n}_0)Y_\pm\pm\mu_\perp^\pm G-i(\Omega\tau)_\pm m\left(\frac{d\tilde{n}_0}{dr}\right)_1\right]\bar{\varphi}_1=0.\end{aligned}\tag{15}$$

In order that we have a nontrivial solution for $\bar{n}_1$ and $\bar{\varphi}_1$ of Eqs. (15), we demand that the determinant of the coefficients must be zero. Hence

$$\begin{aligned}&[-i\omega+X_++\mu_\perp^+F-i(\Omega\tau)_+\mu_\perp^+m(\tilde{E}_0)_1]\Big[-(\tilde{n}_0)Y_--\mu_\perp^-G-\\&i(\Omega\tau)_-\mu_\perp^-m\left(\frac{d\tilde{n}_0}{dr}\right)_1\Big]-[-i\omega+X_--\mu_\perp^-F-i(\Omega\tau)_-\mu_\perp^-m(\tilde{E}_0)_1]\times\\&\left[(\tilde{n}_0)Y_++\mu_\perp^+G-i(\Omega\tau)_+\mu_\perp^+m\left(\frac{d\tilde{n}_0}{dr}\right)_1\right]=0.\end{aligned}\tag{16}$$

Solving for $i\omega$, we obtain

$$-i\omega=\frac{\Lambda+im\Theta}{\chi+im\psi},$$

where

$$\begin{aligned}\Lambda=&-(\tilde{n}_0)(X_+Y_-+X_-Y_+)-(\tilde{n}_0)Fk^2(\mu_\perp^+\mu^--\mu_\perp^-\mu^+)\\&-G(X_+\mu_\perp^-+X_-\mu_\perp^+),\end{aligned}$$

$$\Theta=(\widetilde{E}_0)_1[(\widetilde{n}_0)\{(\Omega\tau)_+\mu_\perp^+Y_-+(\Omega\tau)_-\mu_\perp^-Y_+\}+G\mu_\perp^+\mu_\perp^-\times$$
$$\{(\Omega\tau)_++(\Omega\tau)_-\}]-\left(\frac{dn_0}{dr}\right)_1[(\Omega\tau)_-\mu_\perp^-X_+-(\Omega\tau)_+\mu_\perp^+X_-$$
$$+F\mu_\perp^+\mu_\perp^-\{(\Omega\tau)_++(\Omega\tau)_-\}],$$
$$\chi=(\widetilde{n}_0)(Y_++Y_-)+G(\mu_\perp^++\mu_\perp^-),$$
$$\psi=\left(\frac{dn_0}{dr}\right)_1[(\Omega\tau)_-\mu_\perp^--(\Omega\tau)_+\mu_\perp^+].$$

Solving for the real and imaginary parts for ω from Eq. (17), we obtain

$$\omega_R=\frac{m(\Lambda\psi-\Theta\chi)}{\chi^2+m^2\psi^2},$$
$$\omega_I=\frac{\Lambda\chi+m^2\Theta\psi}{\chi^2+m^2\psi^2}. \tag{18}$$

So now we have obtained the oscillating frequency ω_R and the growth or damping rate characterized by ω_I for each excited mode. The condition for stability is thus determined by ω_I, however, a general treatment of Eq. (18) is quite difficult. In next section, some special cases shall be considered, and a number of conclusions may be reached.

IV. SOME CRITERIA FOR INSTABILITY

Before our discussions, let's first evaluate the following terms. Since

$$G=\frac{2}{d}m^2\int_0^d\frac{n_0}{r}\sin^2\frac{\pi x}{d}dx-\frac{2}{d}\int_0^d\frac{dn_0}{dr}\frac{d}{dx}\left(\sin\frac{\pi x}{d}\right)\sin\frac{\pi x}{d}dx$$
$$-\frac{2}{d}\int_0^d\frac{n_0}{r}\frac{d}{dx}\left(\sin\frac{\pi x}{d}\right)\sin\frac{\pi x}{d}dx$$

upon the integration by parts on the second and third integrands, the above reduces to

$$G=\frac{2}{d}\int_0^d\left(m^2\frac{n_0}{r^2}+\frac{1}{2}\frac{d^2n_0}{dr^2}+\frac{1}{2}\frac{1}{r}\frac{dn_0}{dr}-\frac{1}{2}\frac{n_0}{r^2}\right)\sin^2\frac{\pi x}{d}dx$$
$$=\frac{2}{d}\left(m^2-\frac{1}{2}\right)\int_0^d\frac{n_0}{r^2}\sin^2\frac{\pi x}{d}dx.$$

hence $G>0$, where we note that the combination of the second. and third terms vanishes. And since

$$F=\frac{2}{d}\int_0^d\frac{E_0}{r}\sin^2\frac{\pi x}{d}dx+\frac{2}{d}\int_0^d\frac{dE_0}{dr}\sin^2\frac{\pi x}{d}dx$$
$$+\frac{2}{d}\int_0^d E_0\frac{d}{dx}\left(\sin\frac{\pi x}{d}\right)\sin\frac{\pi x}{d}dx$$
$$=\frac{2}{d}\int_0^d\left(\frac{E_0}{r}+\frac{1}{2}\frac{dE_0}{dr}\right)\sin^2\frac{\pi x}{d}dx.$$

By the substitution of $E_0=C_3/n_0r$, where the sign of C_3 depends on the sign of the applied potential V_0, i. e., C_3 is positive (negative) when the electric field is directing from the center (outer) to the outer (center), we obtain

$$F=\frac{C_3}{d}\int_0^d\left(\frac{1}{n_0r^2}-\frac{dn_0/dr}{n_0^2r}\right)\sin^2\frac{\pi x}{d}dx.$$

It is obvious that the second term inside the bracket is positive definite because of $dn_0/dr<0$, so the sign of F is cooreponding to the sign of the sign of C_3, or V_0. We shall consider some special cases depending on the nature of the density variation and the applied potential.

(A) No Applied Potential

In this case $V_0=0$, so that $F=0, (\widetilde{E}_0)=0$, the numerator of ω_I reduces to

$$-[(\widetilde{n}_0)(X_+Y_-+X_-Y_+)+G(X_+\mu_\perp^-+X_-\mu_\perp^+)][(\widetilde{n}_0)(Y_++Y_-)+G(\mu_\perp^++\mu_\perp^-)]$$
$$-m^2\left(\frac{\widetilde{dn_0}}{dr}\right)[(\Omega\tau)_-\mu_\perp^--(\Omega\tau)_+\mu_\perp^+][(\Omega\tau)_-\mu_\perp^-X_+-(\Omega\tau)_+\mu_\perp^+X_-]. \quad (19)$$

The first term is negative definite. As to the second term, Simon [7] discussed the quantities inside the bracket by taking the limit of large and weak fields, and considering the upper bound of this term compared with the first term, he concluded that Eq. (19) is always negative. In fact, as it is evidented from the numerical values by the substitution of the experimental data, the second term is still negative. Hence the system is stable.

(B) Reversed Field Necessity

We now return to the general result of Eq. (18), the numerator of ω_I takes the form

$$-[(\widetilde{n}_0)(X_+Y_-+X_-Y_+)+(\widetilde{n}_0)Fk^2(\mu_\perp^+\mu^--\mu_\perp^-\mu^+)+G(X_+\mu_\perp^-+X_-\mu_\perp^+)]\times$$
$$[(\widetilde{n}_0)(Y_++Y_-)+G(\mu_\perp^++\mu)]+m^2(\widetilde{E}_0)\left(\frac{dn_0}{dr}\right)_1[(\Omega\tau)_-\mu_\perp^--(\Omega\tau)_+\mu_\perp^+]\times$$
$$[(\widetilde{n}_0)\{(\Omega\tau)_+\mu_\perp^+Y_-+(\Omega\tau)_-\mu_\perp^-Y_+\}+G\mu_\perp^+\mu_\perp^-\{(\Omega\tau)_++(\Omega\tau)_-\}]$$
$$-m^2\left(\frac{\widetilde{dn_0}}{dr}\right)_1^2[(\Omega\tau)_-\mu_\perp^--(\Omega\tau)_+\mu_\perp^+][(\Omega\tau)_-\mu_\perp^-X_+$$
$$-(\Omega\tau)_+\mu_\perp^+X_-+F\mu_\perp^+\mu_\perp^-\{(\Omega\tau)_++(\Omega\tau)_-\}]. \quad (20)$$

Make use of the following inequalities

$$\mu_\perp^+\mu^->\mu_\perp^-\mu^+,$$
$$(\Omega\tau)_-\mu_\perp^->(\Omega\tau)_+\mu_\perp^+,$$
$$(\Omega\tau)_-\mu_\perp^-X_+>(\Omega\tau)_+\mu_\perp^+X_-,$$

it is obvious that the first and third terms of Eq. (20) are negative definite if $F>0$. For the case $F>0$, it means that the applied electric field is directing outward from the center, i.e., $E_0dn_0/dr<0$, and hence the second term is also negative. Hence the system is stable for $E_0dn_0/dr<0$. On the other hand, if $E_0dn_0/dr>0$, i. e., the electric field is directing inward to the center, then $F<0$. Hence there are the following terms. which constribute positive to ω_I and are thus of destabilizing in character:

$$-(\widetilde{n}_0)Fk^2(\mu_\perp^+\mu^- - \mu_\perp^-\mu^+)[(\widetilde{n}_0)(Y_+ + Y_-) + G(\mu_\perp^+ + \mu_\perp^-)$$

$$+ m^2(\widetilde{E}_0)\left(\frac{d\widetilde{n}_0}{dr}\right)_1[(\Omega\tau)_-\mu_\perp^- - (\Omega\tau)_+\widetilde{\mu_\perp^+}][(n_0)\{(\Omega\tau)_+\mu_\perp^+ Y_-$$

$$+ (\Omega\tau)_-\mu_\perp^- Y_+\} + G\mu_\perp^+\mu_\perp^-\{(\Omega\tau)_+ + (\Omega\tau)_-\}] - m\left(\frac{d\widetilde{n}_0}{dr}\right)_1^2 F\times$$

$$\mu_\perp^+\mu_\perp^-\{(\Omega\tau)_+ + (\Omega\tau)_-\}[(\Omega\tau)_-\mu_\perp^- - (\Omega\tau)_+\mu_\perp^+].$$

Equating the about destabilizing terms and the remaining stabilizing terms in Eq. (20), we can then obtain the critical potential. Therefore, we conclude that the necessary condition for instability is $E_0 dn_0/dr > 0$.

The necessity for instability could be understood by considering the physical picture as shown in Fig. 2. An applied electric field $\boldsymbol{E}_0$, directing toward the center, causes the particles drifting in the $\boldsymbol{E}_0 \times \boldsymbol{B}$ direction (O-direction). In a weakly ionized plasma, collisions between the charged particles and the background neutral particles cause the ions and electrons drifting in different speed. The electrons drift more rapidly than the ions do, and then the charge separation produces an induced electric field $\boldsymbol{E}'$ in the B-direction. Subsequently, there produces a drift resulting from $\boldsymbol{E}' \times \boldsymbol{B}$ in the r-direction, which is antiparallel to the direction of the density gradient $r\nabla n_0$, and thus the density distribution is destroyed, the system is unstable.

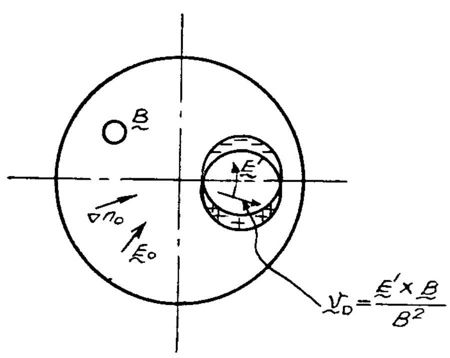

Fig. 2. Instability mechanism.

V. COMPARISON WITH EXPERIMENTS

We shall now check the oscillation frequencies of the excited modes in comparison with Saito's experiments[3]. The configuration of the test tube used is shown in Fig. 3.

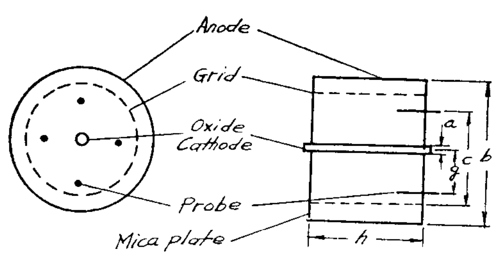

Fig 3. Structure of the test tube.

If the test gas is Argon. and the dimension of electrods (cm) are as follows: $a=0.35$, $b=6.0$, $g=1.5$, $c=5.0$, and $h=5.0$. A space between the anode and the floating grid is occupied by an anode glow, and that between the grid and the cathode by a dark plasma in which four Longuir probes are placed for detecting electron temperature, plasma density, and sigals of oscillations. The results are shown in Fig. 4, and Fig. 5.

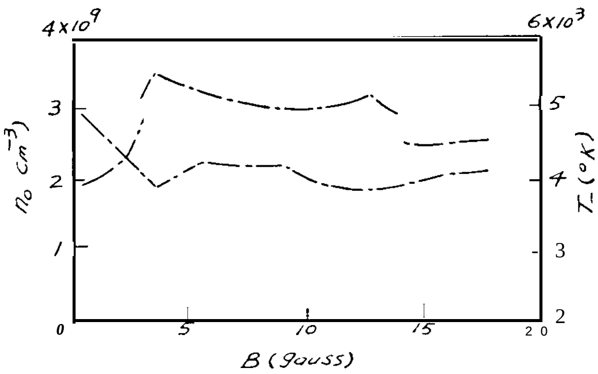

Fig. 4. Electron density and temperature as a function of magnetic field intensity. Discharge current: 20 mA.

In the calculation below, the following numerical values are used: $m_+/m_- = 1.4 \times 10^{-5}$. $T_- = 4{,}000°K$, $T_+ = 300°K$, $\nu_- = 9.2 \times 10^7$ sec, $\nu_+ = 2.8 \times 10^6$ sec. Remember that we need the numerical vaules of n_b in Eq. (8) and V_0 in Eq. (9) to fulfill our calculation. The n_b can be determined simply by numerical substitution. And V_0 will be determined by the following simple approximate formula $-V_0 = (I/A\sigma)d$, where I is the discharge current, A is the area of the tube, and σ is the coefficient of electrical conductivity which is defined by $\sigma = n_0 e^2/m_- \nu_-$. We then substitute all the required numerical values in Eq. (18) to find the oscillating frequency ω_R. The detail calculation is referred to Ref. (7). Now, we summarize our result in Fig. 5.

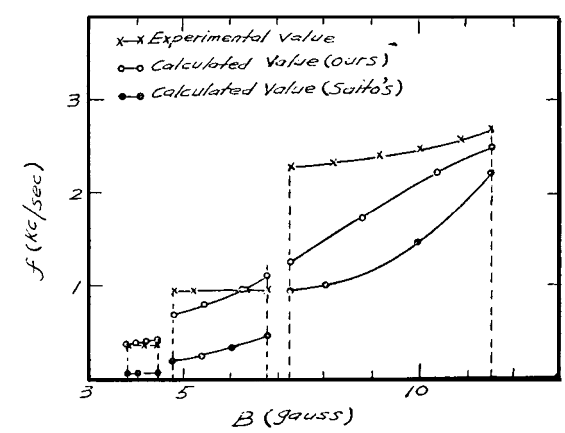

Fig. 5. Frequency as a function of magnetic field intensity under constant discharge current.

(7) C. S. Ma, M. S. thesis, Dept. of Mech. Engineering, National Cheng Kung University (1574).

It is shown that our theoretical results fit the experimental results much better than Saito's do. The reason is clear since our consideration of more realistic model and calculation is more compatible to the existing physical experiment. We would also like to point here the rightness of our assumption of negligible inertia. In Eqs. (2), the ratio of the inertia terms and the collision terms are in the order of $O(\omega/\nu\pm)$, it is readily seen that they are in the order of $O(10^{-3})$ for the ions and of $O(10^{-4})$ for the electrons, which are negligibly small. Hence the assumption of small inertia is verified.