A Reference for the Gravitational Hamiltonian Boundary Term

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The Hamiltonian for physical systems and dynamic spacetime geometry generates the evolution of a spatial region along a vector field. It includes a boundary term which not only determines the value of the Hamiltonian, but also, via the boundary term in the variation of the Hamiltonian, the boundary conditions. The value of the Hamiltonian comes from its boundary term; it gives the quasi-local quantities: energy-momentum and angular-momentum/center-of-mass momentum. This boundary term depends not only on the dynamical variables but also on their reference values; these reference values determine the ground state—the state having vanishing quasi-local quantities. Here our concern is with how to select on the quasi-local two-boundary the reference value. To determine the “best matched” reference metric and connection values for our preferred boundary term for Einstein’s general relativity, we propose on the boundary two-surface (i) four dimensional isometric matching, and (ii) extremizing the value of the energy.

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I. INTRODUCTION

Our objective is to find a good reference for the Hamiltonian boundary term. An important application is to the (quasi-)localization of energy. It is appropriate to first briefly review some aspects of this topic, especially since energy plays a major role in our strategy.

Energy-momentum is the source of gravity (not just for Einstein’s general relativity (GR) but for quite general gravity theories, in particular those that explain gravity in terms of spacetime geometry). Gravitating physical systems have an energy-momentum density, and they can exchange energy-momentum with the gravitational field. This interaction happens locally, and in the process energy-momentum is conserved, nevertheless there is

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no well defined proper local energy-momentum density for geometric gravity itself. This inescapable conclusion (which may seem somewhat ironic—especially since gravity uniquely detects the local density of energy-momentum due to all other physical sources) was actually established already by Noether in the same paper in which she proved her two famous theorems regarding the role of symmetry in dynamical systems [1, 2]; in fact the main motivation for her investigation was to clarify the nature of gravitational energy-momentum and its conservation.

This key feature—which can be understood physically as following from the equivalence principle (for a discussion, see MTW [3], Section 20.4)—explains why standard approaches aimed at identifying an energy-momentum density for gravitating systems always led to various non-covariant, reference frame dependent, energy-momentum complexes (generally referred to as pseudotensors). There are two types of ambiguity. First, there was no unique expression, but rather many (including the well known ones found by Einstein [4], Papapetrou [5], Landau-Lifshitz [6], Bergmann-Thompson [7], Møller [8], Goldberg [9], and Weinberg [10]), so which expression should be used? And second—since all of these expressions are inherently reference frame dependent—for a chosen expression which reference frame should be used to give the proper physical energy-momentum localization?

The more modern idea is quasi-local, i.e., energy-momentum should be associated not with a local density but rather with a closed 2-surface (for a comprehensive review of this topic see [11]).

One particular approach to quasi-local energy-momentum is via the Hamiltonian (the generator of time evolution). It has been shown that the Hamiltonian approach actually includes all the classical pseudotensors as special cases, while taming their inherent problems by providing clear physical and geometric meaning for the two aforementioned ambiguities [12, 13].

II. THE COVARIANT HAMILTONIAN FORMULATION RESULTS

Our research group has developed a covariant Hamiltonian formalism that is applicable to a large class of geometric gravity theories [14–19]. For such theories the Hamiltonian 3-form \( \mathcal{H}(Z) \)—the generator of the evolution of a spatial region along the spacetime displacement vector field \( Z \)—is also a conserved Noether current:

\[
d\mathcal{H}(Z) \propto \text{field eqns} \simeq 0. \tag{1}
\]

It has the general form

\[
\mathcal{H}(Z) = Z^\mu \mathcal{H}_\mu + dB(Z), \tag{2}
\]

where the 3-form \( Z^\mu \mathcal{H}_\mu \)—which generates the evolution equations—is, as a consequence of local diffeomorphism invariance, itself proportional to certain field equations (initial value constraints) and thus vanishes “on shell”. Consequently the value of the Hamiltonian
associated with a spatial region $\Sigma$ is determined by the total differential (boundary) term:

$$E(Z, \Sigma) := \int_{\Sigma} H(Z) = \oint_{\partial \Sigma} B(Z).$$  \hfill (3)

Since it depends only on the field values on the boundary $S = \partial \Sigma$, this value is quasi-local. With suitable choices of the vector field, it can determine values for the quasi-local energy-momentum and angular momentum/center-of-mass momentum.

Note that the boundary 2-form $B(Z)$ can be modified in any way without destroying the conservation property. (This is a particular case of the usual Noether conserved current ambiguity: essentially one can always add a kind of “curl” to a conserved current; this changes the conserved value but does not change the conserved property.) With this freedom one can arrange for almost any conserved quasi-local values. Fortunately the Hamiltonian’s dynamical role tames that freedom.

One must give consideration to the boundary term in the variation of the Hamiltonian (see [20–22]). Requiring it to vanish yields the boundary conditions. The Hamiltonian is functionally differentiable only on the phase space of fields satisfying these boundary conditions. Modifying the boundary term changes the boundary conditions.

Some time ago it was found that each of the “superpotentials” associated with the classical pseudotensors can serve as the Hamiltonian boundary term for GR. Thus each pseudotensor corresponds to a Hamiltonian which evolves the dynamical variables with certain “built in” boundary conditions [12]. The differing boundary conditions physically accounts for their differing energy-momentum values. A similar remark can be made for many of the more modern quasi-local proposals. Fixing the boundary conditions resolves the first type of ambiguity mentioned above.

Looking more closely into the Hamiltonian boundary term, one must, in general, also introduce into it certain reference values which represent the ground state of the field—the “vacuum” (or background field) values. For any quantity $\alpha$ we let $\bar{\alpha}$ be the reference value. Our boundary expression will contain terms of the form $\Delta \alpha := \alpha - \bar{\alpha}$. Here our concern is how to best select these reference values for GR.

III. PREFERRED BOUNDARY TERM FOR GR

For GR two covariant-symplectic boundary terms [15] had been identified; one (which was also found at about the same time by Lynden-Bell, Katz, and Bičák [23] via a Noether argument using a global reference) is our preferred choice:

$$B(Z) = \frac{1}{2\kappa} \left( \Delta \Gamma^\alpha_{\beta} \wedge i_Z \eta^\beta + \bar{D}_\beta Z^\alpha \Delta \eta^\beta \right),$$  \hfill (4)

where $\Gamma^\alpha_{\beta}$ is the connection one-form, $\eta^\alpha\beta\ldots := *(\vartheta^\alpha \wedge \vartheta^\beta \wedge \cdots)$, $i_Z$ denotes the interior product (or contraction) with the vector field $Z$, and $\kappa = 8\pi G/c^4$. This choice corresponds to fixing the orthonormal coframe $\vartheta^\mu$ (equivalently the metric) on the boundary (this follows since the total differential term in the variation of the Hamiltonian 3-form is $d i_Z (\Delta \Gamma^\alpha_{\beta} \wedge$
\[ \delta \eta_{\alpha \beta} \]. At spatial infinity (4) gives appropriate expressions for the energy, momentum, angular-momentum, and center-of-mass momentum [3, 22, 24–27]. (This is not so special, a large class of other expressions can do this also.) The special virtues of the above expression include: (i) at null infinity it directly gives the Bondi-Trautman energy and the Bondi energy flux [18], (ii) it is “covariant”, (iii) it is positive—at least for spherical solutions [28] and large regions, (iv) for small spheres it gives a positive multiple of the Bel-Robinson tensor [19], (v) it yields the first law of thermodynamics for black holes [16], (vi) for spherically symmetric solutions it has the hoop property [28, 29], (vii) for a suitable choice of reference it vanishes for Minkowski space.

IV. THE REFERENCE AND THE QUASI-LOCAL QUANTITIES

For all other fields it is appropriate to choose vanishing reference values as the reference ground state—the vacuum. But for geometric gravity the standard ground state is Minkowski geometry which has a non-vanishing metric, so a non-trivial reference is essential. Minkowski geometry is our chosen reference, but we still need to specify exactly which Minkowski geometry.

To explicitly construct a reference, choose, in a neighborhood of the desired spacelike boundary 2-surface \( S \), four smooth functions \( y^i, i = 0, 1, 2, 3 \) with \( dy^0 \wedge dy^1 \wedge dy^2 \wedge dy^3 \neq 0 \); these quasi-Minkowski coordinates define a Minkowski reference by

\[
\bar{g} = -(dy^0)^2 + (dy^1)^2 + (dy^2)^2 + (dy^3)^2.
\]

(5)

Geometrically, this is equivalent to finding a diffeomorphism embedding a neighborhood of the 2-surface into Minkowski space. The associated reference connection is the pullback of the flat Minkowski connection:

\[
\bar{\Gamma}^\alpha_{\beta \gamma} = x^\alpha_i (\bar{\Gamma}^i_{\gamma \beta} + dy^i_{\beta}) = x^\alpha_i dy^i_{\beta}.
\]

(6)

Here \( x^\alpha_i \) is the inverse of \( y^i_{\alpha} \), where \( dy^i = y^i_{\alpha} dx^\alpha \).

A Killing field of the reference has the infinitesimal Poincaré transformation form \( Z^k = \alpha^k + \lambda^k y^i \), where the translation parameters \( \alpha^k \) and the boost-rotation parameters \( \lambda_{kl} = \lambda_{[kl]} \) are constants. For any chosen reference the 2-surface integral of the Hamiltonian boundary term gives

\[
E(Z, S) = \oint_S B(Z) = -\alpha^k p_k(S) + \frac{1}{2} \lambda_{kl} J^{kl}(S),
\]

(7)

which thus yields both a quasi-local energy-momentum and a quasi-local angular momentum/center-of-mass momentum. As long as the reference approaches at an appropriate rate the flat Minkowski space at spatial infinity, the integrals \( p_k(S) \), \( J^{kl}(S) \) in the spatial asymptotic limit will agree with accepted expressions for these quantities [3, 22, 24–27].

For energy-momentum one takes \( Z \) to be a translational Killing field of the Minkowski reference. Then the second term in our quasi-local boundary expression (4) vanishes. (It also vanishes for 4D isometric matching on \( S \), a condition that we shall use below).
With \( Z^k = \alpha^k = \) constant our quasi-local expression now takes the form

\[
B(Z) = \alpha^k x^\mu_k (\Gamma^\alpha_{\beta \gamma} - x^\alpha_j dy^j_{\beta \gamma}) \wedge \eta_{\mu \alpha \beta}.
\]  

(8)

To explicitly determine the specific values of the quasi-local quantities one needs some good way to choose the reference. For this purpose Minkowski spacetime is the natural choice, especially for asymptotically flat spacetimes. (For other applications a different geometry may be more suitable as a reference, preferably one with high symmetry, e.g., (Anti-)de Sitter; for certain purposes one might choose Friedmann-Lemaître-Robertson-Walker (FLRW) or Schwarzschild).

However, as noted above, almost any four functions will determine some Minkowski reference. With such freedom one can still get almost any value for the quasi-local quantities. This freedom is the quasi-local version of the second type of ambiguity mentioned in the introduction.

Recently we proposed a program [30] to fix the “best” choice of reference. It has two parts: 4D isometric matching and optimization of a certain quantity. Here we present it in more detail—along with a promising alternative optimization. We have already found that our new procedure works well for an important special case: a certain class of axisymmetric spacetimes [31].

V. ISOMETRIC MATCHING OF THE 2-SURFACE

We first recall an important procedure that has been used: isometric matching of the 2-surface \( S \). This can be expressed in terms of quasi-spherical foliation adapted coordinates \( t, r, \theta, \varphi \) as

\[
g^{\prime AB} := g_{AB} - y^0_A y^0_B = -y^0_A y^0_B + \delta_{ab} y^a_A y^b_B,
\]  

(9)

where \( S \) is given by constant values of \( t, r, \) and \( A, B \) range over \( \theta, \varphi \). We use \( \overset{\prime}{} \) to indicate a relation which holds only on the 2-surface \( S \). Eq. (9) is 3 conditions on the 4 functions \( y^a \). One can regard \( y^0 \) as the free choice. From a classic closed 2-surface into \( \mathbb{R}^3 \) embedding theorem—as long as \( S \) and \( y^0(x^\mu) \) are such that on \( S \)

\[
g^{\prime AB} := g_{AB} + y^0_A y^0_B
\]  

(10)

is convex—one has a unique embedding. Wang and Yau have discussed in detail this type of embedding of a 2-surface into Minkowski controlled by one function in their recent quasi-local work [32].

VI. COMPLETE 4D ISOMETRIC MATCHING

Our “new” proposal is: complete 4-dimensional isometric matching on \( S \). (This was already suggested by Szabados back in 2000 at a workshop in Hsinchu, Taiwan; he has
extensively explored this idea [33]). This condition includes more than the aforementioned 2-surface isometric matching, it imposes 10 constraints,

\[ g_{\mu\nu} = \bar{g}_{\mu\nu} = \bar{g}_{ij} \dot{y}^i \dot{y}^j, \]

(11)
on the 16 \( y^\alpha_i(t_0, r_0, \theta, \varphi) \) on \( S \). On the 2-surface \( S \) these 16 quantities are actually determined by 12 independent embedding functions: \( y^t, y^r, y^\theta, y^\varphi \) (since from \( y^i \) on \( S \) one can get \( y^i_\theta, y^i_\varphi \)), hence there remain 2 = 12 – 10 degrees of freedom in choosing the reference. In detail (11) includes: (i) the 3 components of the 2D subsector already considered in (9), which for a given \( y^0 \) determines \( y^3_A \), plus (ii) 7 additional algebraic constraints on the 8 \( y^t, y^r \). More specifically this algebraic system has effectively 2 quadratic and 5 linear constraints. We select \( y^0_r \) as an independent controlling function (geometrically it controls a local boost of the reference in the plane normal to the 2-surface \( S \)). Requiring the existence of suitable algebraic solutions to the 7 off-surface components of (11) imposes some restrictions on the allowable controlling functions, \( y^0, y^0_r \). The sign choices in selecting the appropriate solution of the quadratic relations can be resolved by considering the limiting case of a flat dynamic metric.

One could as an alternative use orthonormal frames. Then the 4D isometric matching can be represented by \( \vartheta^\alpha = \bar{\vartheta}^\alpha \). But the reference coframe has the form \( \bar{\vartheta}^\alpha = dy^\alpha \). Thus one should Lorentz transform the coframe \( \vartheta^\alpha \) to match \( dy^\alpha \) on the 2-surface \( S \). This leads to an integrability condition: the 2-forms \( d\vartheta^\alpha \) should vanish when restricted to the 2-surface:

\[ d\vartheta^\alpha |_S = 0, \]

(12)
this is 4 conditions restricting the 6 parameter local Lorentz gauge freedom. Which again shows that after 4D isometric matching there remains 2 = 6 – 4 degrees of freedom in choosing the reference.

VII. THE BEST MATCHED REFERENCE GEOMETRY

There are 12 embedding variables subject to 10 4D isometric matching conditions (or equivalently, 6 local Lorentz gauge parameters subject to 4 frame embedding conditions). To fix the remaining 2, one can regard the quasi-local value as a measure of the difference between the dynamical and the reference boundary values. This value will be a functional of the 2 reference controlling functions \( y^0, y^0_r \). The critical points of this functional determine the distinguished choices for these 2 functions.

Previously we proposed [30] taking the optimal “best matched” embedding as the one which gives an extreme value to the associated invariant mass \( m^2 = -p_i p_j \bar{g}^{ij} \). This should determine the reference up to a Poincaré transformation.

In principle this is a reasonable condition, but in practice not so practical. The invariant mass is a sum of 4 terms, each quadratic in an integral over \( S \). Note, however, that using the Poincaré freedom one can get the same \( m \) value in the center-of-momentum frame from \( p_0 \). This leads us to our new proposal: take the preferred reference as one that
gives a critical value to the quasi-local energy given by (7) and (8) with $Z^k = \alpha^k = \delta_k^0$. We expect this much simpler optimization to give the same reference geometry as that obtained from using $m^2$.

Based on some physical and practical computational arguments it seems reasonable to expect a unique solution in general. In a numerical calculation in principle one could (i) just calculate the energy values given by (7) and (8) with $Z^k = \alpha^k = \delta_k^0$ for a great many choices of $y_0^0, y_0^1$ subject to the 4D isometric matching constraint (11) and the integrability condition $y^i_A = \partial_A y^i$, and (ii) then note the energy critical points.

Analytically the procedure is complicated by the lack of an explicit formula for the general solution of the 2D isometric embedding. For certain cases with special symmetry this is not an obstacle. This “best matching” procedure already gave reasonable quasi-local energy results for spherically symmetric systems [34], and we now have sensible results for certain axisymmetric systems including the Kerr metric [31].

VIII. CONCLUDING DISCUSSION

Our objective was just to find a good way to select the reference for the Hamiltonian boundary term for GR. Naturally this leads to values for the quasi-local quantities. Moreover, the program has additional benefits, since the results of the construction can be applied to other unanticipated ends. These include: (a) a preferred coordinate frame for the Freud superpotential associated with the Einstein pseudotensor [35] (holonomically—with vanishing reference connection coefficients—the first term in (4) reduces to Freud’s superpotential for the Einstein pseudotensor; thus we are in effect here making a proposal for good coordinates for the Einstein pseudotensor), (b) a preferred tetrad for the teleparallel gauge current [36], (c) an optimal spinor field for the spinor Hamiltonian quasi-local boundary term [37] associated with the Witten positive energy proof [38], and (d) the “best” frame and spinor for the quadratic spinor Lagrangian formulation [39].

Furthermore, from a consideration of the covariant Hamiltonian works [12, 14–19], we can see that our reference program—*isometric embedding with critical energy value*—can be used in other ways, e.g., one can look to the choices of the embedding variables that are critical points of $E(Z, S)$ (7) for a given $Z$. Also it can be applied in much more general settings; such applications include: (i) selecting the reference frame for any of the traditional pseudotensors, and (ii) the reference for the other GR boundary terms corresponding to GR Hamiltonians with other boundary conditions. Indeed the program can be applied to all of the different boundary terms that have been proposed for the most general metric-affine gravity theory [40] and all its special subcases, including the Poincaré gauge theory [41] and teleparallel theory. We have just given a discussion of the Poincaré gauge theory reference choice in [2]. We plan to present detailed discussions for these other applications in future works.
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