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New Commutation Relations for Quantum Gravity

Chopin Soo[1,*] and Hoi-Lai Yu[2,†]

[1]*Department of Physics, National Cheng Kung University, Taiwan*
[2]*Institute of Physics, Academia Sinica, Taiwan*
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A new set of fundamental commutation relations for quantum gravity is presented. The basic variables are the eight components of the unimodular part of the spatial dreibein and eight $SU(3)$ generators which correspond to Klauder's momentric variables. The commutation relations are not canonical, but they have well defined group theoretical meanings. All fundamental entities are dimensionless; and quantum wave functionals are preferentially selected to be in the dreibein representation.

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I. COMMUTATION RELATIONS OF GEOMETRODYNAMICS AND MOMENTRIC VARIABLES

Successful quantization of the gravitational field remains the preeminent challenge a century after the completion of Einstein's general relativity (GR). In quantum field theories, 'equal time' fundamental commutation relations (CR) — part and parcel of causality requirements — are predicated on the existence of spacelike hypersurfaces. Geometrodynamics bequeathed with a positive-definite spatial metric is the simplest consistent framework to implement them. Expressed in terms of a dreibein, e_{ai}, the spatial metric, $q_{ij} := \delta^{ab}e_{ai}e_{bj}$ is automatically positive-definite if the dreibein is real and non-vanishing, modulo $SO(3, C)$ gauge rotations which leave q_{ij} invariant under these local Lorentz transformations. To incorporate fermions, it is also necessary to introduce the dreibein which is more fundamental than the metric.

In usual quantum theories, the fundamental *canonical* CR, $[Q(\mathbf{x}), P(\mathbf{y})] = i\hbar\delta(\mathbf{x}-\mathbf{y})$, implies $\frac{P}{\hbar}$ is the generator of translations of Q which are also symmetries of the 'free theories' in the limit of vanishing interaction potentials. For geometrodynamics, the corresponding canonical CR is $[q_{ij}(\mathbf{x}), \tilde{\pi}^{kl}(\mathbf{y})] = i\hbar\frac{1}{2}(\delta^k_i\delta^l_j + \delta^l_i\delta^k_j)\delta(\mathbf{x} - \mathbf{y})$. However, neither positivity of the spatial metric is preserved under arbitrary translations generated by conjugate momentum; nor is the 'free theory' invariant under translations when interactions are suppressed, because $G_{klmn}\tilde{\pi}^{kl}\tilde{\pi}^{mn}$ in the kinetic part of the Hamiltonian also contains the DeWitt supermetric [1], G_{ijkl}, which is dependent upon q_{ij}. In quantum gravity, states which are infinitely peaked at the flat metric, or for that matter any particular metric with its corresponding isometries, cannot be postulated ad hoc; consequently, the underlying symmetry

*Electronic address: cpsoo@mail.ncku.edu.tw
†Electronic address: hlyu@phys.sinica.edu.tw

of even the 'free theory' is obscure.

Decomposition of the geometrodynamic degrees of freedom, $(q_{ij}, \widetilde{\pi}^{ij})$, singles out the canonical pair $(\ln q^{\frac{1}{3}}, \widetilde{\pi})$ which commutes with the remaining unimodular $\bar{q}_{ij} := q^{-\frac{1}{3}} q_{ij}$, and traceless $\bar{\pi}^{ij} := q^{\frac{1}{3}}\left(\widetilde{\pi}^{ij} - \frac{1}{3}q^{ij}\widetilde{\pi}\right)$ variables. The Hodge decomposition for compact manifolds yields $\delta \ln q^{\frac{1}{3}} = \delta T + \nabla_i \delta Y^i$, wherein the spatially-independent δT is a three-dimensional diffeomorphism invariant (3dDI) quantity which serves as the intrinsic time interval, whereas $\nabla_i \delta Y^i$ can be gauged away since $\mathcal{L}_{\overrightarrow{\delta N}} \ln q^{\frac{1}{3}} = \frac{2}{3}\nabla_i \delta N^i$. A theory of quantum geometrodynamics dictated by first-order Schrödinger evolution in intrinsic time, $i\hbar \frac{\delta \Psi}{\delta T} = H_{\rm Phys.}\Psi$, and equipped with diffeomorphism-invariant physical Hamiltonian and time-ordering has been discussed and advocated in Refs. [2–4]. The Hamiltonian, $H_{\rm Phys} = \int \frac{\bar{H}(\mathbf{x})}{\beta} d^3\mathbf{x}$, and ordering of the time development operator, $U(T, T_0) = \mathbf{T}\{\exp[-\frac{i}{\hbar}\int_{T_0}^{T} H_{\rm Phys}(T')\delta T']\}$, are 3dDI provided

$$\bar{H} = \sqrt{\bar{\pi}^{ij}\bar{G}_{ijkl}\bar{\pi}^{kl} + \mathcal{V}[q_{ij}]} \tag{1}$$

is a scalar density of weight one; and Einstein's GR (with $\beta = \frac{1}{\sqrt{6}}$ and $\mathcal{V} = -\frac{q}{(2\kappa)^2}[R - 2\Lambda_{\it eff}]$) is a particular realization of this wider class of theories. Natural extensions of GR and the precise quantum Hamiltonian are discussed in Refs. [2, 4].

The Poisson brackets for the barred variables are

$$\begin{aligned} &\{\bar{q}_{ij}(\mathbf{x}), \bar{q}_{kl}(\mathbf{y})\} = 0, \quad \{\bar{q}_{kl}(\mathbf{x}), \bar{\pi}^{ij}(\mathbf{y})\} = P^{ij}_{kl}\,\delta(\mathbf{x}-\mathbf{y}), \\ &\{\bar{\pi}^{ij}(\mathbf{x}), \bar{\pi}^{kl}(\mathbf{y})\} = \frac{1}{3}(\bar{q}^{kl}\bar{\pi}^{ij} - \bar{q}^{ij}\bar{\pi}^{kl})\delta(\mathbf{x}-\mathbf{y}), \end{aligned} \tag{2}$$

with $P^{ij}_{kl} := \frac{1}{2}(\delta^i_k\delta^j_l + \delta^i_l\delta^j_k) - \frac{1}{3}\bar{q}^{ij}\bar{q}_{kl}$ denoting the traceless projection operator. This set is not strictly canonical. In the metric representation, the implementation of $\bar{\pi}^{kl}$ as traceless, symmetric, and self-adjoint operators is problematic. Remarkably, these difficulties can be cured by passing to the 'momentric variable' (first introduced by Klauder [5]) which is classically $\bar{\pi}^i_j = \bar{q}_{jm}\bar{\pi}^{im}$. In terms of spatial metric and momentric variables, the fundamental CR postulated (from which the classical Poisson brackets corresponding to (2) can be recovered) are then [4]

$$\begin{aligned} &[\bar{q}_{ij}(\mathbf{x}), \bar{q}_{kl}(\mathbf{y})] = 0, \quad [\bar{q}_{ij}(\mathbf{x}), \bar{\pi}^k_l(\mathbf{y})] = i\hbar \bar{E}^k_{l(ij)}\delta(\mathbf{x}-\mathbf{y}), \\ &[\bar{\pi}^i_j(\mathbf{x}), \bar{\pi}^k_l(\mathbf{y})] = \frac{i\hbar}{2}\left(\delta^k_j\bar{\pi}^i_l - \delta^i_l\bar{\pi}^k_j\right)\delta(\mathbf{x}-\mathbf{y}); \end{aligned} \tag{3}$$

wherein $\bar{E}^i_{j(mn)} := \frac{1}{2}\left(\delta^i_m\bar{q}_{jn} + \delta^i_n\bar{q}_{jm}\right) - \frac{1}{3}\delta^i_j\bar{q}_{mn}$ (with properties $\delta^j_i\bar{E}^i_{j(mn)} = \bar{E}^i_{j(mn)}\bar{q}^{mn} = 0; \bar{E}^i_{jil} = \bar{E}^i_{jli} = \frac{5}{3}\bar{q}_{jl}$) is the vielbein for the supermetric $\bar{G}_{ijkl} = \bar{E}^m_{n(ij)}\bar{E}^n_{m(kl)}$. Quantum mechanically, the momentric operators and CR *can be explicitly realized in the metric representation* by

$$\bar{\pi}^i_j(\mathbf{x}) := \frac{\hbar}{i}\bar{E}^i_{j(mn)}(\mathbf{x})\frac{\delta}{\delta\bar{q}_{mn}(\mathbf{x})} = \frac{\hbar}{i}\frac{\delta}{\delta\bar{q}_{mn}(\mathbf{x})}\bar{E}^i_{j(mn)}(\mathbf{x}) = \hat{\bar{\pi}}^{\dagger i}_j(\mathbf{x}), \tag{4}$$

which are self-adjoint on account of $[\frac{\delta}{\delta\bar{q}_{mn}(\mathbf{x})}, \bar{E}^i_{j(mn)}(\mathbf{x})] = 0$.

II. NEW COMMUTATION RELATIONS

These momentric variables generate $SL(3,R)$ transformations of $\bar{q}_{ij} = \delta^{ab}\bar{e}_{ai}\bar{e}_{bj}$ which preserve its positivity and unimodularity. Moreover, they generate at each spatial point, an $SU(3)$ algebra. In fact, with 3×3 Gell-Mann matrices $\lambda^{A=1,\dots,8}$, it can be checked that $T^A(\mathbf{x}) := \frac{1}{\hbar\delta(\mathbf{0})}(\lambda^A)^j_i\hat{\bar{\pi}}^i_j(\mathbf{x})$ generates the $SU(3)$ algebra with structure constants $f^{AB}{}_C$ [6]. In (3) there is an asymmetry in that there are only five independent components in the symmetric unimodular $\bar{q}_{ij}$ (likewise for the symmetric traceless $\bar{\pi}^{ij}$ in (2)), whereas the mixed-index traceless momentric variable, $\bar{\pi}^i_j$, contains eight components. This asymmetry is rectified by unimodular dreibein-traceless momentric variables, $(\bar{e}_{ai} := e^{-\frac{1}{3}}e_{ai}, T^A)$, each having eight independent components, and they obey the *new fundamental CR advocated in this work*,

$$[\bar{e}_{ai}(\mathbf{x}), \bar{e}_{bj}(\mathbf{y})] = 0, \qquad [\bar{e}_{ai}(\mathbf{x}), T^A(\mathbf{y})] = i(\frac{\lambda^A}{2})^k_i\bar{e}_{ak}\frac{\delta(\mathbf{x}-\mathbf{y})}{\delta(\mathbf{0})},$$
$$[T^A(\mathbf{x}), T^B(\mathbf{y})] = if^{AB}{}_C T^C\frac{\delta(\mathbf{x}-\mathbf{y})}{\delta(\mathbf{0})}. \tag{5}$$

A number of remarkable and intriguing features are encoded in this set of CR. It is noteworthy that in (5) all entities, including $(\bar{e}_{ai}, T^A)$ and $\frac{\delta(\mathbf{x}-\mathbf{y})}{\delta(\mathbf{0})}$, are dimensionless; and *neither the gravitational coupling constant nor Planck's constant make their appearance.* $\delta(\mathbf{0}) := \lim_{x\to y}\delta(\mathbf{x}-\mathbf{y})$ denotes the coincident limit; so there are no divergences in $\frac{\delta(\mathbf{x}-\mathbf{y})}{\delta(\mathbf{0})}$ which is unity in the coincident limit and vanishing otherwise. The second CR in (5) implies

$$\exp\left(i\int\alpha_B T^B\delta(\mathbf{0})d^3\mathbf{y}'\right)\bar{e}_{ai}(\mathbf{x})\exp\left(-i\int\alpha_A T^A\delta(\mathbf{0})d^3\mathbf{y}\right) = \left(\exp(\frac{\alpha_A(\mathbf{x})\lambda^A}{2})\right)^j_i\bar{e}_{ja}(\mathbf{x}), \tag{6}$$

with $\exp(\frac{\alpha_A(\mathbf{x})\lambda^A}{2})$ being a local $SL(3,R)$ transformation; while the final CR in (5) is the statement that $T^A(\mathbf{x})$ generates, at each spatial point, a separate $SU(3)$ algebra.

Local Lorentz transformations are generated by the Gauss Law constraint $G_a(\mathbf{x}) := \epsilon_{abc}e^b_i\tilde{\pi}^{ci} - i\tilde{\pi}_\psi\frac{\tau_a}{2}\psi = 0$, wherein classically $\tilde{\pi}^{ai} := 2\tilde{\pi}^{ij}e^a_j$ is the canonical conjugate momentum of the dreibein, $(\psi, \tilde{\pi}_\psi)$ denotes the conjugate pair of Weyl fermions, and τ_a are the Pauli matrices. However, $e := \det(e_{ai})$ and $(\ln q^{\frac{1}{3}}, \tilde{\pi})$ are Lorentz singlets which commute with the Gauss Law constraint, and the latter can be reexpressed with traceless momentric, rather than momentum, variable as $G_a = \epsilon_{abc}\bar{e}^b_i\bar{e}^{cj}\bar{\pi}^i_j - i\tilde{\pi}_\psi\frac{\tau_a}{2}\psi = 0$. That it generates $SO(3,C)$ local Lorentz rotations of the dreibein can be verified from $[\bar{e}_{ai}(\mathbf{x}), \frac{i}{\hbar}\int\eta^b G_b d^3\mathbf{y}] = \epsilon_{abc}\eta^b(\mathbf{x})\bar{e}^c_i(\mathbf{x})$, with η^b as gauge parameter; in addition, it generates the associated local $SL(2,C)$ transformations of the fermionic degrees of freedom (d.o.f.).

The generators T^A obey the $SU(3)$ algebra of (5) and thus do not totally commute among themselves. A remarkable consequence is that quantum wave functions cannot be chosen as functionals of T^A, but quantization in the dreibein representation is viable and preferentially selected by the fundamental CR. This differs starkly from the usual *canonical*

CR which allows wave functionals to be realized in either of the conjugate representations; and it may explain why our universe seems fundamentally and intuitively 'metric' in nature, and not 'conjugately realized'. The dreibein (and metric representation) consistently guarantees that hypersurfaces on which the fundamental CR are defined will be spacelike. An explicit realization in the dreibein representation is $T^A(\mathbf{x}) = -\frac{i}{\delta(\mathbf{0})}(\frac{\lambda^A}{2})^j_k \bar{e}_{ja}(\mathbf{x})\frac{\delta}{\delta \bar{e}_{ka}(\mathbf{x})}$; and when operating on wave functionals of the spatial metric, $\Psi[q_{ij}]$, this action of T^A agrees with that of (4).

The symmetry of the free theory now becomes transparent. It is characterized by $SU(3)$ invariance generated by the momentric, because the Casimir invariant T^AT^A is related to the kinetic operator through

$$\frac{\hbar^2[\delta(\mathbf{0})]^2}{2}T^AT^A = \bar{\pi}^{i\dagger}_j\bar{\pi}^j_i = \bar{\pi}^i_j\bar{\pi}^j_i = \bar{\pi}^{ij}\bar{G}_{ijkl}\bar{\pi}^{kl}. \quad (7)$$

The spectrum of the free Hamiltonian can thus be labeled by eigenvalues of the complete commuting set at each spatial point comprising the two Casimirs $L^2 = T^AT^A, C = d_{ABC}T^AT^BT^C \propto \det(\bar{\pi}^i_j)$ with $d_{ABC} = \frac{1}{4}Tr(\{\lambda_A, \lambda_B\}\lambda_C)$, Cartan subalgebra T^3, T^8, and isospin $I = \sum_{B=1}^3 T^BT^B$. An underlying group structure has the advantage that the action of momentric on wave functionals by functional differentiation can be traded for the well defined action of generators of $SU(3)$ on states expanded in this basis. To wit,

$$(\frac{\lambda^A}{2})^j_k \frac{\bar{e}_{ja}(\mathbf{x})}{i\delta(\mathbf{0})}\frac{\delta}{\delta\bar{e}_{ka}(\mathbf{x})}\langle\bar{e}_{bl}|\prod_y|l^2, C, I, m_3, m_8\rangle_y = \langle\bar{e}_{bl}|T^A(\mathbf{x})\prod_y|l^2, C, I, m_3, m_8\rangle_y. \quad (8)$$

Group theoretical considerations also lead to a succinct description of graviton d.o.f. and their associated quantum excitations. In geometrodynamics, local $SL(3,R)$ transformations of $\bar{q}_{kl}$ are generated through $U^\dagger(\alpha)\bar{q}_{kl}(\mathbf{x})U(\alpha) = (e^{\frac{\alpha(\mathbf{x})}{2}})^m_k\bar{q}_{mn}(\mathbf{x})(e^{\frac{\alpha(\mathbf{x})}{2}})^n_l$, wherein $U(\alpha) = e^{-\frac{i}{\hbar}\int \alpha^i_j\bar{\pi}^j_i d^3\mathbf{y}}$ [5]; while the generator of spatial diffeomorphisms for the momentric and unimodular spatial d.o.f. is effectively $D_i = -2\nabla_j\bar{\pi}^j_i$, with smearing $\int N^iD_id^3\mathbf{x} = \int(2\nabla_jN^i)\bar{\pi}^j_i d^3\mathbf{x}$ after integration by parts. The action of spatial diffeomorphisms can thus be subsumed by specialization to $\alpha^i_j = 2\nabla_jN^i$, with the upshot that $SL(3,R)$ transformations which are *not* spatial diffeomorphisms are parametrized by α^i_j complement to $2\nabla_jN^i$. Given a background metric $q^{\mathcal{B}}_{ij} = q^{\frac{1}{3}}\bar{q}^{\mathcal{B}}_{ij}$, this complement is precisely characterized by the choice of transverse traceless (TT) parameter $(\alpha_{TT})^i_j := q^{\mathcal{B}}_{jk}\alpha^{(ik)}_{Phys}$, because the condition $\nabla^j_B(\alpha_{TT})^i_j = 0$ excludes non-trivial N^i through $\nabla^2_{\mathcal{B}}N^i = 0$ if $(\alpha_{TT})^i_j$ were of the form $2\nabla^{\mathcal{B}}_jN^i$. The TT conditions impose four restrictions on the symmetric $\alpha^{(ij)}_{Phys}(\mathbf{x})$, leaving exactly two free parameters. The action of $U_{Phys}(\alpha_{TT}) = e^{-\frac{i}{\hbar}\int(\alpha_{TT})^i_j\bar{\pi}^j_i d^3\mathbf{x}}$ (which is thus local $SL(3,R)$ modulo spatial diffeomorphism) on any 3dDI wave functional would result in an inequivalent state. TT conditions however require a particular background metric to be defined. In Ref. [7] a basis of infinitely squeezed states was explicitly realized by Gaussian wave functionals $\Psi[\bar{q}]_{q^{\mathcal{B}}} \propto \exp[-\frac{1}{2}\int \tilde{f}_\epsilon(\bar{q}_{ij} - \bar{q}^{\mathcal{B}}_{ij})\bar{G}^{ijkl}_{\mathcal{B}}(\bar{q}_{kl} - \bar{q}^{\mathcal{B}}_{kl})d^3\mathbf{x}]$. 3dDI is recovered in the limit of zero Gaussian width with divergent $\lim_{\epsilon\to 0}\tilde{f}_\epsilon \to \delta(\mathbf{0})$.

These localized Newton-Wigner states are infinitely peaked at q^B_{ij} which can then be deployed to actualize the TT conditions. The action of $U_{Phys}(\alpha_{TT})$ on these states would thus generate two infinitesimal local physical excitations at each spatial point.

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