Analytical Study of an MLC Circuit with Quasiperiodic Forcing

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An Murali-Lakshmanan-Chua circuit is the simplest second order non-autonomous nonlinear circuit exhibiting chaos. The strange nonchaotic attractor behaviour of the circuit when subjected to quasiperiodic forcing is studied analytically. An explicit analytical solution for a system exhibiting strange nonchaotic attractor behaviour in its dynamics is presented here for the first time, with phase portraits showing the Heagy-Hammel and fractalization routes to chaos.

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I. INTRODUCTION

Strange nonchaotic attractors (SNA) are regarded as structures in between regularity and chaos. They are geometrically strange as evidenced by their fractal structure, which is common to all chaotic systems. However, they are nonchaotic in a dynamical sense, because they do not show sensitivity with respect to initial conditions (as evidenced by negative Lyapunov exponents), just like regular systems. Following the initial study of Grebogi et al. [1], several theoretical as well as experimental studies have been made pertaining to the existence and characterization of SNAs in different quasiperiodically driven nonlinear dynamical systems. Among the various possible routes leading to chaos through the evolution of SNAs, the Heagy-Hammel or torus doubling route to chaos and the fractalization of the torus are the prominent routes. The existence of SNAs in simple nonlinear electronic circuits has been extensively studied both experimentally and numerically [5–8].

The Murali-Lakshmanan-Chua (MLC) circuit is a simple second-order dissipative circuit, with a Chua diode as the nonlinear element, which was suggested by Murali et al. [2]. The period doubling dynamics of the circuit leading to chaotic motion was studied both experimentally and numerically [2, 4]. An explicit analytical solution to the normalized circuit equations of the MLC circuit were presented [4, 9, 10]. The MLC circuit has a piecewise linear nonlinear element, the Chua diode, which is linear within the three regions. Since the circuit equations describing the MLC circuit is a second order differential equation, the equation is solved for each linear region of the Chua diode. Similar solutions were given to some second order nonlinear chaotic circuits with piecewise nonlinear element as the active circuit component [11–14]. The analytical solutions thus obtained were used to explain the dynamics of the circuit through phase portraits of the state variables.

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The different routes to chaos through SNAs in second-order nonautonomous electronic circuits were studied both experimentally and numerically by Venkatesan et al. [4], Thamilmaran et al. [5], Senthikumar et al. [6], and Srinivasan et al. [7]. Studies on strange nonchaotic attractors are mostly through numerical experiments and are not studied analytically. In the present work we give an explicit analytical solution to the MLC circuit subjected to quasiperiodic sinusoidal forcing that exhibits SNAs in its dynamics. The resulting solution is plotted to get phase portraits, which reveals the Heagy-Hammel and fractalization routes to chaos.

II. CIRCUIT EQUATIONS

The circuit consists of a simple series LCR network, forced by two sinusoidal voltage generators $F_1 \sin(\omega_1 t)$ and $F_2 \sin(\omega_2 t)$ with the Chua’s diode $N$ parallel to the capacitor $C$ acting as the nonlinear element is as shown in Fig. 1. By applying Kirchhoff's voltage and current laws the voltage $v$ across the capacitor $C$ and the current $i_L$ through the inductor $L$ are are given by the following set of two first-order coupled nonautonomous differential equations.

\begin{align}
C \frac{dv}{dt} &= i_L - g(v), \\
L \frac{di_L}{dt} &= -R_i i_L - v + F_1 \sin(\omega_1 t) + F_2 \sin(\omega_2 t),
\end{align}

where $R_i = R + R_a$, and $g(v)$ is the piecewise linear function given by

\begin{equation}
g(v) = G_b v + 0.5(G_a - G_b)[|v + B_p| - |v - B_p|],
\end{equation}
where \( G_a = -0.76 \) mS, \( G_b = -0.41 \) mS and \( B_p = 1.0 \) V are the respective values of the negative slopes of the inner and outer regions and the breakpoints in the \((v-i)\) characteristic curve of the nonlinear element. In the above equation \((1b)\), if either of the forces \( F_1 \) or \( F_2 \) is zero, then the circuit is the simple MLC circuit possessing a rich variety of dynamics which are well studied both experimentally and numerically \([2, 4]\). If both the forces \( F_1 \) and \( F_2 \) are zero then the dynamics is only a fixed point. When the two forces \( F_1 \) and \( F_2 \) exist then different routes to chaos of the system exhibiting SNAs are observed \([5]\). The circuit parameters take the values \( C = 10 \) nF, \( L = 18 \) mH, \( R_s = 20 \) Ω, \( R = 1440 \) Ω, \( \omega_1 = 23706.6 \) Hz, and \( \omega_2 = 7325.7 \) Hz.

### III. EXPLICIT ANALYTICAL SOLUTIONS

In this section, we will investigate the analytical aspects of this quasiperiodically forced series LCR circuit with Chua’s diode as the nonlinear element. Eqs. \((1)\) are written in the dimensionless form as

\[
\begin{align*}
\dot{x} &= y - g(x), \\
\dot{y} &= -\beta y - \nu \beta y - \beta x + f_1 \sin(\omega_1 t) + f_2 \sin(\omega_2 t),
\end{align*}
\]

where, \( \beta = C/LG^2, \nu = GR_s, f_1 = F_1/\beta B_p, f_2 = F_2/\beta B_p, a = G_a/G, b = G_b/G, \) and \( G = 1/R. \) The piecewise linear function \( g(x) \) is given by

\[
g(x) = bx + 0.5(a - b)[|x + 1| - |x - 1|] 
\]

or

\[
g(x) = \begin{cases} 
  bx + (a - b), & \text{if } x \geq 1, \\
  ax, & \text{if } |x| \leq 1, \\
  bx - (a - b), & \text{if } x \leq -1.
\end{cases}
\]

The values of the rescaled quantities are \( \beta = 1.152, \nu = 0.01388, \omega_1 = 2.1448, \) and \( \omega_2 = 0.6627. \) The analytical solution can be found by solving the set of normalized equations given in Eqs. \((3)\), for the chosen circuit parameters used in the experiment. From the \((v-i)\) characteristic of the nonlinear element, we have \( a = -1.09440, b = -0.5904, \) and \( \beta = 1.152, \) associated with the regions \( D_0, D_+, \) and \( D_- \).

One can easily establish that a unique equilibrium point \((x_0, y_0)\) exists for Equation \((3)\) in each of the following three subsets:

\[
\begin{align*}
D_+ &= \{(x, y)|x > 1\}P^+ = (-k_1, -k_2), \\
D_0 &= \{(x, y)|x < 1\}O = (0, 0), \\
D_- &= \{(x, y)|x < -1\}P^- = (k_1, k_2),
\end{align*}
\]

where \( k_1 = (a - b)\sigma/\beta + b\sigma, \) \( k_2 = \beta(b - a)/\beta + b\sigma, \) and \( \sigma = \beta + \nu\beta. \) The stability of the fixed points given in Equation \((6)\), can be calculated from the stability matrices.
Region $D_0$

In this region, the stability determining eigenvalues are calculated from the stability matrix

$$J_0 = \begin{pmatrix} -a & 1 \\ -\beta & -\sigma \end{pmatrix},$$

where $\sigma = \beta + \nu \beta$, and the eigenvalues are obtained as $\alpha_1 = 0.320430226$, $\alpha_2 = -0.394030215$. From the eigenvalues, we infer that the fixed point corresponding to the region $D_0$ is a saddle or hyperbolic equilibrium point.

Region $D_\pm$

The stability determining eigenvalues for this region are calculated from the stability matrix

$$J_\pm = \begin{pmatrix} -b & 1 \\ -\beta & -\sigma \end{pmatrix},$$

where $\sigma = \beta + \nu \beta$, and the eigenvalues are found to be a pair of complex conjugates given as $\alpha_1 = -0.288799999 + i(0.615635736)$, $\alpha_2 = -0.288799999 - i(0.615635736)$ and the fixed point corresponding to the $D_\pm$ region is a stable focus equilibrium point.

Naturally, these fixed points can be observed depending upon the initial conditions $x(0)$ and $y(0)$ of Eqs. (3) when $f_1 = f_2 = 0$. In fact, Eqs. (3) can be integrated explicitly in terms of elementary functions in each of the three regions $D_0$, $D_\pm$, and the resulting solutions can be matched across the boundaries to obtain the full solution as given below.

The solution of that equation is given by $[x(t; x_0, y_0), y(t; x_0, y_0)]^T$ for which the initial condition is written as $(t, x, y) = (t_0, x_0, y_0)$. Since Eq. (1) is piecewise linear, the solution in each of the three regions can be obtained explicitly as follows.

III-1. The central region: $D_0(|x| \leq 1)$

In this region, $g(x) = ax$ where, ‘$a$’ represents the normalized slope of the $(v - i)$ characteristic curve and hence Eqs. (3a) and (3b) become

$$\dot{x} = y - ax,$$

$$\dot{y} = -\sigma y - \beta x + f_1 \sin(\omega_1 t) + f_2 \sin(\omega_2 t).$$

Differentiating Eq. (9b) with respect to time and using Eqs. (9a, 9b) in the resultant equation, we obtain

$$\ddot{y} + Ay + By = f_1 a \sin(\omega_1 t) + f_1 \omega_1 \cos(\omega_1 t) + f_2 a \sin(\omega_2 t) + f_2 \omega_2 \cos(\omega_2 t),$$

where

$$A = \sigma + a, \quad B = \beta + a \sigma.$$  

The roots of Equation (6) are given by

$$\alpha_{1,2} = \frac{-(A) \pm \sqrt{(A^2 - 4B)}}{2}. \quad (11)$$
The roots of the Equation (6) are real and unequal. The general solution of Equation (6) is written in the form of a linear second order inhomogeneous differential equation with constant coefficients as

\[ y(t) = C_1 e^{\alpha_1 t} + C_2 e^{\alpha_2 t} + E_1 + E_2 \sin \omega_1 t + E_3 \cos \omega_1 t + E_4 \sin \omega_2 t + E_5 \cos \omega_2 t, \]

where \( C_1, C_2, E_1, E_2, E_3, E_4, \) and \( E_5 \) are constants to be fixed from the initial \((x_0, y_0)\) conditions and are given by

\[
C_1 = \frac{e^{-\alpha_1 t_0}}{\alpha_2 - \alpha_1} \left((\sigma + \alpha_2)y_0 + \beta x_0 - \alpha_2 E_1\right) + ((E_2\omega_1 - E_3\alpha_2) \cos \omega_1 t_0) \\
- (f_1 + E_3\omega_1 + E_2\alpha_2) \sin \omega_1 t_0 \sin \omega_2 t_0) \\
- (f_2 + E_5\omega_2 + E_4\alpha_2) \sin \omega_2 t_0, \\
\]

\[
C_2 = \frac{e^{-\alpha_2 t_0}}{\alpha_1 - \alpha_2} \left((\sigma + \alpha_1)y_0 + \beta x_0 - \alpha_1 E_1\right) + ((E_2\omega_1 - E_3\alpha_1) \cos \omega_1 t_0) \\
- (f_1 + E_3\omega_1 + E_2\alpha_1) \sin \omega_1 t_0 \sin \omega_2 t_0) \\
- (f_2 + E_5\omega_2 + E_4\alpha_1) \sin \omega_2 t_0. \\
\]

\[ E_1 = 0, \]

\[ E_2 = \frac{(f_1 \omega_1^2 (A - a) + a f_1 B)}{A^2 \omega_1^2 + (B - \omega_1^2)^2}, \]

\[ E_3 = \frac{(f_1 \omega_1 (B - aA - \omega_1^2))}{A^2 \omega_1^2 + (B - \omega_1^2)^2}, \]

\[ E_4 = \frac{(f_2 \omega_2^2 (A - a) + a f_2 B)}{A^2 \omega_2^2 + (B - \omega_2^2)^2}, \]

\[ E_3 = \frac{(f_2 \omega_2 (B - aA - \omega_2^2))}{A^2 \omega_2^2 + (B - \omega_2^2)^2}. \]

By substituting the value of the arbitrary constants \( C_1, C_2, E_1, E_2, E_3, E_4, E_5, \) the explicit form of the solution \( x(t) \) can be obtained straightaway from Eq. (3b) as

\[ x(t) = \left(\frac{1}{\beta}\right) (f_1 \sin(\omega_1 t) + f_2 \sin(\omega_2 t) - \dot{y} - \sigma y). \]

III-2. The outer region: \( D_+(x \geq 1) \)

In the outer region \( D_+ \) the characteristic function is chosen as \( g(x) = bx + (a - b) \), where ‘\( b \)’ is related to the slope of the \((v - i)\) characteristic of the nonlinear element in that region. On substituting the above values of \( g(x) \) in Eq. (3a), we obtain

\[ \dot{x} = y - bx - (a - b), \]

\[ \dot{y} = -\sigma y - \beta x + f_1 \sin(\omega_1 t) + f_2 \sin(\omega_2 t). \]
Differentiating Eq. (21b) with respect to time and using Eqs. (21a, 21b) in the resultant equation, we obtain

\[ \ddot{y} + Ay + By = f_1 b \sin(\omega_1 t) + f_2 b \sin(\omega_2 t) + f_2 \omega_2 \cos(\omega_2 t) + \Delta, \] (22)

where

\[ A = \sigma + b, \quad B = \beta + b\sigma, \quad \Delta = \beta(a - b). \]

The roots of Eq. (18) are complex conjugates given by

\[ \alpha_1 = u + iv, \quad \alpha_2 = u - iv, \]

where

\[ u = -\frac{A}{2}, \quad v = \frac{\sqrt{4B - A^2}}{2}. \]

The general solution for Eq. (18) is then given as

\[ y(t) = e^{ut}(C_3 \cos(vt) + C_4 \sin(vt)) + E_6 + E_7 \sin \omega_1 t + E_8 \cos \omega_1 t + E_9 \sin \omega_2 t + E_0 \cos \omega_2 t + \Delta, \] (23)

where \( C_3, C_4, E_6, E_7, E_8, E_9, \) and \( E_0 \) are constants to be fixed from the initial conditions \((x_0, y_0)\) and are given by

\[
C_3 = e^{-ut_0} \left( \frac{(\sigma + u)y_0 + \beta x_0 - uE_6)}{v} \sin vt_0 + v(y_0 - E_6) \cos vt_0 + ((E_7 \omega_1 - uE_8) \sin vt_0 - vE_8 \cos vt_0) \cos \omega_1 t_0 
- ((f_1 + E_8 \omega_1 + uE_7) \sin vt_0 + vE_7 \cos vt_0) \sin \omega_1 t_0 
+ ((E_9 \omega_2 - uE_0) \sin vt_0 - vE_0 \cos vt_0) \cos \omega_2 t_0 
- ((f_2 + E_0 \omega_2 + uE_9) \sin vt_0 + vE_9 \cos vt_0) \sin \omega_2 t_0, \right) \]

\[
C_4 = e^{-ut_0} \left( \frac{(\sigma + u)y_0 - \beta x_0) \cos vt_0 + v(y_0 - E_6) \sin vt_0}{v} \right) 
- ((E_7 \omega_1 - uE_8) \cos vt_0 + vE_8 \sin vt_0) \cos \omega_1 t_0 
+ ((f_1 + E_8 \omega_1 + uE_7) \cos vt_0 - vE_7 \sin vt_0) \sin \omega_1 t_0 
- ((E_9 \omega_2 - uE_0) \cos vt_0 + vE_0 \sin vt_0) \cos \omega_2 t_0 
+ ((f_2 + E_0 \omega_2 + uE_9) \cos vt_0 - vE_9 \sin vt_0) \sin \omega_2 t_0, \right) \] (24)
\[ E_6 = \frac{\Delta}{B}, \quad (26) \]
\[ E_7 = \frac{(f_1 \omega_1^2 (A - b) + bf_1 B)}{A^2 \omega_1^2 + (B - \omega_1^2)^2}, \quad (27) \]
\[ E_8 = \frac{(f_1 \omega_1 (B - bA - \omega_1^2))}{A^2 \omega_1^2 + (B - \omega_1^2)^2}, \quad (28) \]
\[ E_9 = \frac{(f_2 \omega_2^2 (A - b) + bf_2 B)}{A^2 \omega_2^2 + (B - \omega_2^2)^2}, \quad (29) \]
\[ E_0 = \frac{(f_2 \omega_2 (B - bA - \omega_2^2))}{A^2 \omega_2^2 + (B - \omega_2^2)^2}. \quad (30) \]

By substituting the value of the arbitrary constants \( C_3, C_4, E_6, E_7, E_8, E_9, E_0 \), the explicit form of the solution \( x(t) \) can be obtained from Eq. (3b) as

\[ x(t) = \left( \frac{1}{\beta} \right) (f_1 \sin(\omega_1 t) + f_2 \sin(\omega_2 t) - \dot{y} - \sigma y). \quad (31) \]

**III-3. The outer region \( D_- (x \leq 1) \)**

In this region \( D_+ \) the characteristic function is chosen as \( g(x) = bx - (a - b) \). On substituting the above values of \( g(x) \) in Eq. (3a), we obtain

\[ \dot{x} = y - bx + (a - b), \quad (32a) \]
\[ \dot{y} = -\sigma y - \beta x + f_1 \sin(\omega_1 t) + f_2 \sin(\omega_2 t). \quad (32b) \]

Differentiating Eq. (32b) with respect to time and using Eqs. (32a, 32b) in the resultant equation, we obtain

\[ \ddot{y} + Ay + By = f_1 b \sin(\omega_1 t) + f_1 \omega_1 \cos(\omega_1 t) + f_2 b \sin(\omega_2 t) + f_2 \omega_2 \cos(\omega_2 t) + \Delta, \quad (33) \]

where

\[ A = \sigma + b, \quad B = \beta + b \sigma, \quad \Delta = \beta(b - a). \]

Eq. (29) is similar to that of Eq. (18), except for \( \Delta \) taking the value \( \Delta = \beta(b - a) \). The general solution for this region is given as

\[ y(t) = e^{ut}(C_3 \cos(vt) + C_4 \sin(vt)) + E_6 + E_7 \sin(\omega_1 t) + E_8 \cos(\omega_1 t) + E_9 \sin(\omega_2 t) + E_0 \cos(\omega_2 t) + \Delta. \quad (34) \]

The arbitrary constants \( C_3, C_4, E_6, E_7, E_8, E_9, E_0 \), assume the same values as in \( D_+ \) and \( x(t) \) is obtained as

\[ x(t) = \left( \frac{1}{\beta} \right) (f_1 \sin(\omega_1 t) + f_2 \sin(\omega_2 t) - \dot{y} - \sigma y). \quad (35) \]
FIG. 2: Heagy-Hammel route to chaos. (a) 1-Torus: $f_2 = 0.18$, (b) 2-Tori: $f_2 = 0.22$, (c) SNA: $f_2 = 0.2357$, (d) Chaos: $f_2 = 0.2377$.

FIG. 3: Fractalization route to chaos. (a) 3-Tori: $f_2 = 0.335$, (b) SNA: $f_2 = 0.4531$, (c) Chaos: $f_2 = 0.4585$. 
Now let us briefly explain how the solution can be generated in the \((x - y)\) phase space. Thus if we start with the initial conditions \(x(t = 0) = x_0, y(t = 0) = y_0\) in the region \(D_0\) region at time \(t = 0\), the arbitrary constants \(C_1\) and \(C_2\) get fixed. Then \(x(t)\) evolves as given in Eq. (20), up to either \(t = T_1\), when \(x(T_1) = 1\) and \(\dot{x}(T_1) > 0\) or \(t = T'_1\), when \(x(T'_1) = -1\) and \(\dot{x}(T'_1) < 0\). Knowing whether \(T_1 < T'_1\) or \(T_1 > T'_1\) we can determine the next region of interest \((D_{\pm})\), and the arbitrary constants of the solutions of that region can be fixed by matching the solutions. The procedure can be continued for each successive crossing. In this way, the explicit solutions can be obtained in each of the regions \(D_0, D_{\pm}\). However, it is clear that sensitive dependence on initial conditions is introduced in each of these crossings at appropriate parameter regimes during the inverse procedure of finding \(T_1, T'_1, T_2, T'_2, \ldots\), etc. from the solutions.

Using the above analytical solutions the phase portraits of the circuit equations can be drawn to show the Heagy-Hammel and fractalization routes to chaos through strange nonchaotic attractors. The amplitude of the first force \(f_1\) is kept constant at a value of \(f_1 = 0.5184\), while the amplitude of the second force \(f_2\) is varied to get the torus doubling route and fractalization of the torus. The Heagy-Hammel route to chaos is as shown in Fig. 2 through SNA. With a further increase in the amplitude \(f_2\), the fractalization of a three tori is observed. Fig. 3 shows the fractalization route to chaos through the evolution of SNA.

**IV. CONCLUSION**

In this paper, we presented the analytical results of the dynamics of a second order nonautonomous dissipative nonlinear circuit subjected to a quasiperiodic force. The SNA behavior of chaotic circuits are studied both numerically and experimentally but not studied analytically. An explicit analytical solution for a system exhibiting a strange nonchaotic attractor behaviour is reported here for the first time.

**References**


