Dirac Bracket for Pedestrians

M. K. Fung

Department of Physics, National Taiwan Normal University, Taipei, Taiwan 116, Republic of China
(Received May 20, 2014)

We present an exposition explaining the implication of the Dirac bracket as well as its application to derive equations of motion in the full phase space for some simple dynamical systems in classical mechanics.

DOI: 10.6122/CJP.20140609 PACS numbers: 04.20.Cv, 45.20.Jj

I. INTRODUCTION

The Dirac bracket is a highbrow mathematical tool in theoretical physics. It was introduced by Dirac [1] for the purpose of doing proper quantization procedures for constrained systems. It was immediately applied to gauge and gravitational theories. This notion is rarely discussed in quantum mechanics, save that recently in the book of Weinberg [2]. In the realm of classical mechanics there is a short section on the Dirac problem in the article by Arnold [3]. However, on examination it will be found that the notion of the Dirac bracket, thus presented, is buried in a labyrinth of formal manipulation and generalization. The true value of the Dirac bracket is hidden as one just realizes the beauty of the resulting formula. To delve out its more elegant implication we proceed in the following with some simple worked examples in classical mechanics. Side by side with the other ways of solving the same problem we shall be able to appreciate the elegance of this formulation in a better way.

This article is organized as follows. After this introduction we start in Section II with a study of the most basic constrained dynamical system, namely a free particle moving in a circle. We deal with the Lagrangian formulation here. There are two treatments presented. One is to solve for the constraint and to describe the system solely in terms of the angular variable $\theta$. The other is to consider the full space, but with the introduction of a Lagrange multiplier we can do free variation in the extended space. In Section III we make a transition to the Hamiltonian formulation. The most important concept is to consider the Poisson bracket which defines the symplectic structure in the evolution of the dynamical variables. We introduce a definition of the reduced Poisson bracket so that the Poisson bracket is defined in terms the reduced phase space only. We can also consider the usual Poisson bracket at the expense of the inclusion of a Lagrange multiplier. In Section IV we usher in the notion of the Dirac bracket. In this case we work in the full phase space with the elimination of the Lagrange multiplier. The constraint equations are embedded into the symplectic structure so that the Dirac bracket of the constraints with any dynamical variable is zero. We would say that we have a degenerate symplectic structure in the full
phase space. The fundamental Dirac brackets for the dynamical problems are depicted as well as the evolution equations. It turns out that the differential equations are quite simple, exhibiting the symmetry of the constraints. In Section V we elaborate on more complicated dynamical systems, breaking the circular symmetry, anticipating elliptic function solutions for the equations of motion. Finally in Section VI we make a conclusion.

II. LAGRANGIAN FORMULATION

The sole object of this article is to study constrained systems in classical mechanics. For pedagogical reasons we give here a detailed study of a simple case—that is a free particle in two dimensions constrained to move in a circle. The Lagrangian of a free particle in two dimensions is given by

\[ L = \frac{1}{2} (\dot{x}^2 + \dot{y}^2), \tag{1} \]

where the mass \( m \) is set to unity. The constraint equation is

\[ \phi(x, y) = x^2 + y^2 - 1 = 0, \tag{2} \]

where we only consider the geometrical constraint denoted by \( \phi \), which is a function of the coordinates only.

There are two ways to deal with this problem, and the two ways are totally different in concept. The direct way is to solve for the constraint equation getting the right number of degrees of freedom. In this case, this is doable with the substitution \( x = \cos \theta \) and \( y = \sin \theta \). The Lagrangian reduces to \( L = \frac{1}{2} \dot{\theta}^2 \) giving the equation of motion as \( \ddot{\theta} = 0 \). The solution is \( x = \cos(\omega t + \delta) \) and \( y = \sin(\omega t + \delta) \), where \( \omega^2 = \dot{x}^2 + \dot{y}^2 \), which is a constant of motion, and \( \delta \) is the arbitrary phase for the initial conditions. So this is a uniform circular motion.

The second way is a more flexible one. Instead, it extends the degrees of freedom with a Lagrange multiplier \( \lambda \) [4]. The variables \( x \) and \( y \) can be freely varied with the modified Lagrangian

\[ L' = \frac{1}{2} (\dot{x}^2 + \dot{y}^2) - \lambda (x^2 + y^2 - 1), \tag{3} \]

For the least action principle, we do get the equations of motion

\[ \dot{x} = -2\lambda x, \quad \dot{y} = -2\lambda y, \quad 0 = x^2 + y^2 - 1. \tag{4} \]

Manipulating these equations we can get \(-2\lambda = x\dot{x} + y\dot{y}\). But the third constraint equation gives \( x\ddot{x} + y\ddot{y} = 0 \) and \( x\dddot{x} + y\dddot{y} = -(\dot{x}^2 + \dot{y}^2) \) on differentiation, giving

\[ 2\lambda = x^2 + y^2, \tag{5} \]
which is a constant of motion by the use of the equations of motion. Now we can put
\[ \omega = \pm \sqrt{2\lambda} \]
where the sign depends on the initial conditions. So we get \[ x = \cos(\omega t + \delta) \] and \[ y = \sin(\omega t + \delta) \]. It should be remarked that in this simple case \( \lambda \) turns out to be a constant, but generically the Lagrange multiplier can be a complicated function of the coordinates and velocities. In plain language we can say that we employ a different mechanical system of a particle attached to a spring with suitable strength and appropriate initial conditions to simulate a uniform circular motion.

III. HAMILTONIAN FORMULATION

We shall now go for the Hamiltonian description of this dynamical system. The Hamiltonian formalism is a good transit to the canonical formalism of the quantum theory. The Hamiltonian for this simple example is just
\[ H = \frac{1}{2} (p_x^2 + p_y^2), \]
where the canonical conjugate momenta are given as \( p_x = \dot{x} \) and \( p_y = \dot{y} \) and we work in the phase space of the dynamical system. The constraint equation is still \( x^2 + y^2 = 1 \).

This is a system with constraints and the usual textbook procedure would not work. Like in the previous section we shall adopt two different ways to treat this problem. First of all we shall take the easy route of reducing the problem in the new variable \( \theta \). The conjugate momentum will be found out to be \( p_{\theta} = \dot{\theta} \) with the hamiltonian in the simple form \( H = \frac{1}{2} p_{\theta}^2 \).

Time evolution in the phase space is succinctly described by the Hamiltonian with the well-known Poisson bracket. The Poisson bracket for the coordinate \( \theta \) and its conjugate momentum \( p_{\theta} \) is given as \( \{ \theta, p_{\theta} \} = 1 \). The evolution equations are \( \dot{\theta} = \{ \theta, H \} = p_{\theta} \), and \( \dot{p}_{\theta} = \{ p_{\theta}, H \} = 0 \) giving \( p_{\theta} = \omega \) a constant such that \( \omega^2 = (\dot{x}^2 + \dot{y}^2) \), and \( \theta = \omega t + \delta \).

At this stage it is instructive to introduce a reduced bracket for the \((x, y, p_x, p_y)\) phase space where the usual Poisson bracket calculation does not work because the variables have constraint equations among them. This reduced bracket amounts to reducing the dynamical variables to \((\theta, p_{\theta})\) and working out the Poisson bracket in terms of the reduced variables. The fundamental reduced brackets can be worked out as follows.

\[
\begin{align*}
[x, y] &= 0; \\
[x, p_x] &= \sin^2 \theta; \\
[y, p_x] &= -\sin \theta \cos \theta; \\
[p_x, p_y] &= -\dot{\theta}; \\
[x, p_y] &= -\sin \theta \cos \theta; \\
[y, p_y] &= \cos^2 \theta.
\end{align*}
\]

We do not bother to rewrite the results in terms of the \((x, y, p_x, p_y)\) phase space variables since in the next section we shall present an alternative way of calculation without solving for the constraint equation at all. The equations of motion can also be calculated yielding
the following equations.

\[
\begin{align*}
\dot{x} &= [x, H] = \sin^2 \theta \beta_x - \sin \theta \cos \theta \beta_y , \\
\dot{y} &= [y, H] = -\sin \theta \cos \theta \beta_x + \cos^2 \theta \beta_y , \\
\dot{\beta}_x &= [\beta_x, H] = -\theta \beta_y , \\
\dot{\beta}_y &= [\beta_y, H] = \theta \beta_x .
\end{align*}
\] (8)

This set of equations seems complicated, but it just offers solutions as previously calculated.

Now we can proceed with the alternative approach of not solving the constraint equation by introducing multiplier function. By the usual steps we get the Hamiltonian

\[
H' = \frac{1}{2} (p_x^2 + p_y^2) + \lambda (x^2 + y^2 - 1) ,
\]

(9)
together with the constraint equation \(x^2 + y^2 = 1\). The usual Poisson bracket can be employed in the \((x, y, p_x, p_y)\) phase space since they are freely varying. However there is a subtle point in defining the time evolution equations with the Poisson bracket. We must take the Poisson bracket before imposing the constraint equation, Dirac [1] called this a weak equation denoted by the weak equality \(\approx\). The evolution equation of any dynamical variable \(f\) in the \((x, y, p_x, p_y)\) phase space will be represented by the equation

\[
\dot{f} \approx \{ f, H' \} \approx \{ f, H \} + \lambda \{ f, \phi \} .
\]

(10)

Because of the weak equality definition the Poisson bracket is well-defined even if we have not worked out the expression for the multiplier \(\lambda\) yet. Now the constraint equation should be valid for all time, but its Poisson bracket with \(H'\) equals

\[
\{ \phi, H' \} \approx \{ \phi, H \} \approx 2xp_x + 2yp_y ,
\]

(11)

which is not apparently equal to zero. Indeed, if we solve the evolution equations together with the original constraint, we indeed get \(\dot{\phi} \equiv (xp_x + yp_y) = 0\). The constraint obtained from the compatibility condition for the Poisson bracket with \(H'\) is called the secondary constraint. The equation \(\dot{\phi} = 0\) is called the primary constraint. We do not get more constraints with the compatibility condition if our Lagrangian equations are second order in nature. The equations of evolution will be determined by the Poisson bracket with \(H'\) together with the primary constraint. This formalism is a bit awkward since the equations of evolution contain \(\lambda\) which has to be determined. In the next section we shall discuss a more sophisticated procedure doing without the service of the multiplier \(\lambda\). This involves a new bracket definition, that is, the well-known Dirac bracket.

IV. DIRAC BRACKET UNDER SCRUTINY

To proceed, we define an extended hamiltonian

\[
H'' = \frac{1}{2} (p_x^2 + p_y^2) + \lambda (x^2 + y^2 - 1) + \mu (xp_x + yp_y) .
\]

(12)
This extension is legitimate as we have the freedom of adding a total derivative to the Lagrangian. Now we treat the primary and secondary constraints with equal importance. With respect to this Hamiltonian $H''$ we find that the Poisson brackets of the constraint equations with the Hamiltonian are given by

$$\{\phi, H''\} \approx \{\phi, H\} + \lambda \{\phi, \phi\} + \mu \{\phi', \phi'\},$$

$$\{\phi', H''\} \approx \{\phi', H\} + \lambda \{\phi', \phi\} + \mu \{\phi, \phi'\}.$$  \hspace{1cm} (13)

We have retained the vanishing terms in the equation to make the equations more symmetric. The trick is that we can let these explicitly be equal to zero so as to solve for $\lambda$ and $\mu$. Generically, we get the following results.

$$\begin{pmatrix} \lambda \\ \mu \end{pmatrix} = -\begin{pmatrix} \{\phi, \phi\} \\ \{\phi', \phi'\} \end{pmatrix}^{-1} \begin{pmatrix} \{\phi, H\} \\ \{\phi', H\} \end{pmatrix}.$$ \hspace{1cm} (14)

Specifically for our example, with $\{\phi, \phi'\} = 2(x^2 + y^2) = 2$, the multipliers are solved as

$$\lambda = \frac{1}{2}(p_x^2 + p_y^2),$$

$$\mu = -(xp_x + yp_y),$$

which are consistent with our other computations.

Now we can define the Dirac bracket for the evolution equation as

$$\dot{f} = [f, H]_D \equiv \{f, H''\} = \{f, H\} - \{(f, \phi) \{f, \phi'\}\}^{-1} \begin{pmatrix} \{\phi, \phi\} \\ \{\phi', \phi'\} \end{pmatrix}^{-1} \begin{pmatrix} \{\phi, H\} \\ \{\phi', H\} \end{pmatrix},$$ \hspace{1cm} (17)

where the right hand side of the equation involves only operations on the full $(x, y, p_x, p_y)$ phase space without the multipliers. Specifically, for our example the equations of motion are calculated to be

$$\begin{align*}
\dot{x} &= p_x - x(xp_x + yp_y) = -y(xp_y - yp_x), \\
\dot{y} &= p_y - y(xp_x + yp_y) = x(xp_y - yp_x), \\
\dot{p}_x &= -x(p_x^2 + p_y^2) + px(xp_x + yp_y) = -(xp_y - yp_x)p_y, \\
\dot{p}_y &= -y(p_x^2 + p_y^2) + py(xp_x + yp_y) = (xp_y - yp_x)p_x.
\end{align*}$$ \hspace{1cm} (18)

Upon examination it can be seen that this set of equations is the same as those of Eq. (8). It is not hard to solve these equations of motion if it can be observed that $(xp_y - yp_x)^2 = (p_x^2 + p_y^2)$ by means of the constraint equations. So the angular momentum is a constant of motion. With this as a constant designated by $\omega$ we can see that both $x$ and $y$ and $p_x$ and $p_y$ are separately rotating with angular velocity $\omega$.

In general for a constrained Hamiltonian in the full $(q, p)$ phase space with a set of complete constraints $\phi_i(q, p) = 0$ we can define a reduced symplectic structure with the Dirac bracket for any $f(q, p)$ and $g(q, p)$ as

$$[f, g]_D \equiv \{f, g\} - \{f, \phi_i\} C_{\phi_i, \phi_j}^{-1} \{\phi_j, g\},$$ \hspace{1cm} (19)
where we employ the notation $C_{\phi_i, \phi_j} = \{\phi_i, \phi_j\}$. From the definition it can be seen that $[\phi_k, f]_D = 0$ for any $f(q, p)$. So with the Dirac bracket the constraint equations are explicitly and consistently imposed. Specifically for our example, the fundamental Dirac brackets are given as follows.

\[
\begin{aligned}
[x, y]_D &= 0; & [p_x, p_y]_D &= -xp_y + yp_x; \\
[x, p_x]_D &= 1 - x^2; & [x, p_y]_D &= -xy; \\
[y, p_x]_D &= -yx; & [y, p_y]_D &= 1 - y^2.
\end{aligned}
\]

(20)

So this set of equations is equal to that of the fundamental reduced brackets Eq. (7).

Furthermore it can be observed that for circular motion it is advisable to employ the complex variables $x + iy = e^{i\theta}$ and its complex conjugate $x - iy = e^{-i\theta}$. In terms of these variables the fundamental Dirac brackets take the form

\[
\begin{aligned}
[x + iy, x - iy]_D &= 0; & [p_x + ip_y, p_x - ip_y]_D &= 2i(xp_y - yp_x); \\
[x + iy, p_x - ip_y]_D &= 1; & [x + iy, p_x + ip_y]_D &= -(x + iy)^2; \\
[x - iy, p_x - ip_y]_D &= -(x - iy)^2; & [x - iy, p_x + ip_y]_D &= 1.
\end{aligned}
\]

(21)

The variables $x + iy$ and $p_x - ip_y$ are like co-ordinate and momentum conjugated variables. So are the variables $x - iy$ and $p_x + ip_y$. The other Dirac brackets are a consequence of the constraint equations $x^2 + y^2 = 1$ and $xp_x + yp_y = 0$, demonstrating that the dynamical system is one-dimensional. That the variables $p_x + ip_y$ and $p_x - ip_y$ have nonzero Dirac bracket reminds us of the Poisson bracket for a particle in a magnetic field involving circular motion.

The equations of motion in terms of these complex variables can be easily written down as

\[
\begin{aligned}
\dot{x} + i\dot{y} &= i(xp_y - yp_x)(x + iy), \\
\dot{p}_x + i\dot{p}_y &= i(xp_y - yp_x)(p_x + ip_y).
\end{aligned}
\]

(22)

V. FURTHER EXAMPLES FOR APPLICATION

The reader may cast a doubt, saying that our example studied so far is too simple, and the use of the Dirac bracket formalism may not be so elegant an approach for more complicated cases. So in this section we present further examples to illustrate that this formulation is good as well. Our previous example has equations of motion solved as trigonometric functions. There are more complicated dynamical systems that have elliptic functions as their solutions. We shall make a cursory discussion of such systems which denote different ways of breaking the circular symmetry. One notable example will the case of a simple but rigid pendulum [5]. Here the Hamiltonian is given as

\[
H = \frac{1}{2}(p_x^2 + p_y^2) + g(1 - x),
\]

(23)
where we have taken the potential energy to be zero at $x = 1$, $x$ is the vertically downward direction, and $g$ is the constant gravitational pull. Of course, we have the constraints $\phi = x^2 + y^2 - 1 = 0$ and $\phi' = xp_x + yp_y = 0$.

The fundamental Dirac brackets are the same as those of Eq. (7), since the constraint equations are the same. With a different hamiltonian $H$, the equations of motion will be

\begin{align*}
\dot{x} &= -y(xp_y - yp_x), \\
\dot{y} &= x(xp_y - yp_x), \\
\dot{p}_x &= -(xp_y - yp_x)p_y + gy^2, \\
\dot{p}_y &= (xp_y - yp_x)p_x - gxy.
\end{align*}

(24)

It is evident that we have the energy $E$ as a good conserved constant of motion, i.e.,

\begin{equation}
\frac{1}{2}(p_x^2 + p_y^2) + g(1 - x) = E.
\end{equation}

The angular momentum is not conserved, but is expressible as $(xp_y - yp_x) = \sqrt{2E - 2g(1 - x)}$. The co-ordinate $x(t)$ can be integrated as an elliptic function, and $y(t)$ can be calculated as a result of the constraint equation $x^2 + y^2 = 1$.

It is interesting to note that one can take the square root of the complex variable $\sqrt{x + iy} = \alpha + i\beta$, so that we can employ a change of variables as $x = \alpha^2 - \beta^2$ and $y = 2\alpha\beta$. It comes out that the half angle variable is more convenient in this case. Indeed, here we have the Hamiltonian now defined as

\begin{equation}
\frac{1}{8}(p_\alpha^2 + p_\beta^2) + \frac{1}{2}g\beta^2.
\end{equation}

(26)

\begin{equation}
\frac{1}{2}(p_\alpha^2 + p_\beta^2) + \frac{1}{2}\left(\omega_1^2x^2 + \omega_2^2y^2\right),
\end{equation}

(27)

$\omega_1 \neq \omega_2$ and with the same two constraints $\phi$ and $\phi'$. 

Next in complexity will be the Neumann problem [6] in which we have an anisotropic oscillator constrained to move in a circle. The Hamiltonian reads as

\begin{equation}
\frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}\left(\omega_1^2x^2 + \omega_2^2y^2\right),
\end{equation}

(27)
Again the fundamental Dirac brackets are the same. The equations of motion are

\[
\dot{x} = -y(xp_y - yp_x), \\
\dot{y} = x(xp_y - yp_x), \\
\dot{p}_x = -(xp_y - yp_x)p_y - x y^2(\omega_1^2 - \omega_2^2), \\
\dot{p}_y = (xp_y - yp_x)p_x - yx^2(\omega_1^2 - \omega_2^2). 
\]

(27)

Now \(2E = (p_x^2 + p_y^2) + (\omega_1^2 x^2 + \omega_2^2 y^2)\), and \((xp_y - yp_x) = \sqrt{2E - (\omega_1^2 x^2 + \omega_2^2 y^2)}\). So the co-ordinate variables can be integrated as prototype Jacobian elliptic functions.

Alternatively, we can deform the constraint equation to be a curve of an ellipse, \(\phi = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0\) as well as \(\phi' = \frac{xp_x}{a^2} + \frac{yp_y}{b^2} = 0\), together with the free particle Hamiltonian \(H = \frac{p_x^2}{2} + \frac{p_y^2}{2}\). A little computation gives \(\{\phi, \phi'\} = \frac{2}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4}}}\). So the fundamental Dirac brackets are deformed as

\[
[x, y]_D = 0; \\
[p_x, y]_D = - \frac{(xp_y - yp_x)/a^2b^2}{\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right)}; \\
[x, p_x]_D = \frac{y^2/b^4}{\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right)}; \\
[x, p_y]_D = - \frac{xy/a^2b^2}{\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right)}; \\
[y, p_x]_D = \frac{x^2/a^4}{\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right)}; \\
[y, p_y]_D = \frac{x^2/a^4}{\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right)}. 
\]

(28)

The equations of motion are easily computed to be

\[
\dot{x} = -y \left(\frac{xp_y - yp_x}{b^2}\right) \left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right), \\
\dot{y} = x \left(\frac{xp_y - yp_x}{a^2}\right) \left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right), \\
\dot{p}_x = -p_y \left(\frac{(xp_y - yp_x)/a^2b^2}{\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right)}\right), \\
\dot{p}_y = p_x \left(\frac{(xp_y - yp_x)/a^2b^2}{\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right)}\right). 
\]

(29)
This set of equations seems to be a little bit out of symmetry. However, the constraint equations come to the rescue. It can be observed that we have the equation
\[ \left( \frac{x^2}{a^4} + \frac{y^2}{b^4} \right) \left( p_x^2 + p_y^2 \right) = \left( \frac{xp_y - yp_x}{a^2} \right)^2. \]
This leads to the possible solvability of the equations of motion for the co-ordinates, e.g.,
\[ \dot{x} = -\frac{y}{b^2} \frac{\sqrt{p_x^2 + p_y^2}}{\sqrt{\left( \frac{x^2}{a^4} + \frac{y^2}{b^4} \right)}}, \]
and a similar equation for \( \dot{y} \). Since \( p_x^2 + p_y^2 = 2E \) is a constant of motion, the integration can be done via another kind of elliptic function. We can also show that
\[ \frac{p_x^2}{a^2} + \frac{p_y^2}{b^2} = \left( \frac{xp_y - yp_x}{a^2b^2} \right)^2, \]
a duality between the energy and the angular momentum squared.

VI. DISCUSSION AND CONCLUSION

We hope that our exposition can clearly elucidate the formula of the Dirac bracket, since we have a very simple and basic constrained dynamical system as an example. We have a degenerate symplectic structure in the full phase space. There may be a more mathematical elegant way to reduce the symplectic structure. However, this approach explains that technically the Dirac bracket works so well because the constrained equations are satisfied for all the time if we define time evolution with the help of the Dirac bracket. Although the Dirac bracket was invented for quantizing gauge fields and the gravitational field, one would expect that this notion of Dirac bracket should work equally well for constrained system in quantum theory. The quantum theory of angular momentum is obviously our next object to study via the Dirac bracket.

References