Some Physical Aspects of Moyal Noncommutativity

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We present in this work a new insight towards adapting the famous idea of the Zeeman effect to noncommutativity à la Moyal.

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I. INTRODUCTION

Noncommutativity of coordinates or non-commutative geometry (NCG), is a very old idea [1]. The use of NCG in string theory was initiated for the first time by Witten [2]. Recently, through the original work carried out by Connes, Douglas, and Schwartz [3, 4], this idea took a new direction in mathematics and physics. Later on, several important contributions on NCG followed these seminal works [5–7], we refer the readers to the textbooks [8] for general references on NCG and string theory.

We aim through this work to contribute to this subject by using Moyal noncommutativity [9, 10] and its various applications in 2d integrability and conformal symmetry. We suggest then an interpretation of the noncommutative (sl$_2$ KdV and sl$_3$ Boussinesq)-Burgers mappings [11, 12] by implementing the famous idea of the Zeeman effect.

II. BASIC DEFINITIONS

Let us first start by specifying the nature of the objects used in this work. The functions often involved in the two dimensional phase-space are arbitrary functions which are generally indicate by $f(x, p)$ with coordinates $x$ and $p$. With respect to this phase space, we have to define the following objects:

1. The constants $f_0$ are defined such that

$$\partial_x f_0 = 0 = \partial_p f_0.$$  \hspace{1cm} (1)

2. The functions $u_i(x, t)$’s depending on an infinite set of variables $t_1 = x, t_2, t_3, ..., $ with

$$\partial_p u_i(x, t) = 0.$$  \hspace{1cm} (2)

The index $i$ stands for the conformal weight of the field $u_i(x, t)$. These functions can be considered in the complex language framework as being the analytic (conformal) fields of...
conformal spin \(i = 1, 2, \ldots\).

3. Other objects that we will use are given by
\[
  u_i(x, t) \star p^j.
\]

These are objects of conformal weight \((i + j)\) living on the non-commutative space parameterized by \(\theta\). Throughout this work, we will use the following notation conventions: \([u_i] = i, [\theta] = 0,\) and \([p] = [\partial_x] = −[x] = 1,\) where the symbol \([\ ]\) stands for the conformal dimension of the enclosed object.

4. The star product law defining the multiplication of objects in the non-commutative space has been shown to satisfy the following expression
\[
  f(x, p) \star g(x, p) = \sum_{s=0}^{\infty} \sum_{i=0}^{s} \frac{\theta^s}{s!} (-1)^i c_s^i (\partial_x^i \partial_p^{s-i} f)(\partial_x^{s-i} \partial_p^i g)
\]
with \(c_s^i = \frac{s^i}{s!(s-i)!}\).

5. The Moyal bracket is defined as
\[
  \{f(x, p), g(x, p)\}_\theta = \frac{f \star g - g \star f}{2\theta}.
\]

6. In order to distinguish the classical objects from the \(\theta\)-deformed ones, we consider the following convention notations:
   a) \(\tilde{\Sigma}^{(r,s)}_m\): is the space of momentum Lax differential operators of conformal spin \(m\) and degree \((r, s)\) with \(r \leq s\). Typical operators of this space are given by
\[
  \sum_{i=r}^{s} u_{m-i} \star p^i.
\]
   b) \(\tilde{\Sigma}^{(0,0)}_m\): is the space of coefficient functions \(u_m\) of conformal spin \(m, m \in \mathbb{Z}\), which may depend on the parameter \(\theta\). It coincides in the standard limit, \(\theta = 0\), with the ring of analytic fields involved in the construction of conformal symmetry and \(W\)-extensions.
   c) \(\tilde{\Sigma}^{(k,k)}_m\): is the space of momentum operators of type
\[
  u_{m-k} \star p^k.
\]

7. The \(\theta\)-Leibnitz rules \([9]\)]
\[
  p^n \star f(x, p) = \sum_{s=0}^{n} \theta^s c_n^s f^{(s)}(x, p)p^{n-s},
\]
and the important
\[
  p^{-n} \star f(x, p) = \sum_{s=0}^{\infty} (-)^s \theta^s c_n^{s-1} f^{(s)}(x, p)p^{-n-s},
\]
where \( f'(s) = \partial^s_x f \) is the prime derivative.

8. The Ring of analytic functions:

A convenient description consists in using the complex language notation in which we define the two dimensional Euclidean space parametrized by \( z = t + ix \) and \( \bar{z} = t - ix \). In this notation, the currents of conformal weight \( k \) are simply written as \( u_k(x, t) \equiv u_k(z) \).

It is now the time to introduce the space of analytic functions of arbitrary conformal spin. This is the space of completely reducible infinite dimensional \( so(2) \) Lorentz representations that can be written as

\[
\Sigma^{(0,0)} = \bigoplus_{k \in \mathbb{Z}} \Sigma_k^{(0,0)},
\]

where the \( \Sigma_k^{(0,0)} \)'s are one dimensional \( so(2) \) spin \( k \) irreducible modules. The upper indices \( (0, 0) \) carried by the space \( \Sigma^{(0,0)} \) are special values of the general indices \( (p, q) \) describing the lowest and highest degrees of Lax operators of type \( \sum_{i=p}^q u_{m-i} \star p^i \). The generators belonging to the space \( \Sigma^{(0,0)} \) are given by the spin \( k \) analytic fields. They may be viewed as analytic maps \( u_k \) which associate to each point \( z \) on the unit circle the fields \( u_k(z) \). For \( k \geq 2 \), these fields can be thought of as the higher spin currents involved in the construction of \( w_\theta \)-algebras. A particular example is given by the spin-2 current \( u_2(z) \) intimately associated with the Virasoro algebra \( T(z) \).

9. The classical limit

Since we are interested in the \( \theta \)-deformation case, we have to add that the spaces \( \Sigma_k^{(0,0)} \) are \( \theta \)-dependent; the corresponding \( w_\theta \)-algebra is shown to exhibit new properties related to the \( \theta \)-parameter, and reduces to the standard \( w \)-algebra once some special limits are performed. As an example, consider for instance the \( w_\theta^3 \)-algebra generalizing the Zamolodchikov algebra. The conserved currents of this extended algebra are shown to take the following form:

\[
w_2 = u_2, \quad w_3 = u_3 - \theta u'_2,
\]

which coincides with the classical case once the limit \( \theta = \frac{1}{2} \) is performed. It is the convenient limit than we must consider in field theory and integrable systems to assure compatibility with the extended conformal symmetry (Zamolodchikov algebra) since the standard limit \( \theta = 0 \) doesn’t respect this objective.

III. ZEEMAN EFFECT IN THE FRAMEWORK OF THE BURGERS-\( sl_n \) KDV MAPPING

We present in this section an original approach to interpreting some results previously established that concern the mapping between the NC Burgers integrable system and the NC deformation of the \( sl_n \)-KdV integrable hierarchies in the Moyal momentum space [9–12]. This approach consists in adapting, artfully, the famous idea of the Zeeman effect to the Moyal algebra of NC Lax operators to provide an alternative issue and interpretation of some properties that are encountered in the study of NC integrable hierarchies.
Let us underline in this context that the presence of the deformation parameter $\theta$ in the Moyal algebra is important, in the sense that it can lead us to identify it with the inverse of the magnetic field $B$, as is well known, such that

$$\theta \sim B^{-1}. \quad (12)$$

Before considering such an application, it is useful to recall some essential basic notions of the Zeeman effect, as well as the importance of the magnetic field in this context.

**III-1. Zeeman effect: Basic ideas**

1. **Definitions:**

   The following definitions are equivalent:

   1. It is well known that an atom can be characterized by a unique set of discrete energy states such that, when excited, the atom makes transitions between these quantized energy states and emits light. The emitted light is shown to form a discrete spectrum, reflecting the quantized nature of the energy levels. In the presence of a magnetic field, these energy levels can shift, this is the Zeeman effect.

   2. Analogously to the Stark effect characterized by the splitting of a spectral line into several components in the presence of an electric field, the Zeeman effect is defined as being the splitting of a spectral line into several components in the presence of a magnetic field.

2. **Origin of the Zeeman effect:**

   The origin of the Zeeman effect can be simply presented as follows: Let’s consider an atomic energy state such that an electron orbits around the nucleus of the atom. This electron has a magnetic dipole moment associated with its angular momentum. In the presence of a magnetic field, the electron acquires an additional energy and consequently the original energy level is shifted. The energy shift may be positive, zero, or even negative, depending on the angle between the electron magnetic dipole moment and the field.

**III-2. Zeeman effects from the mapping**

We have shown in previous works [11, 12] how it’s possible to establish a correspondence between the $sl_n$-KdV NC integrable hierarchies and the NC Burgers system. This issue of mapping exhibits a particular interest because it allows us to install the first steps toward a possible unification of NC integrable models in the Moyal momentum framework.

Presently, we are interested in studying another aspect in relation with the Zeeman effect. The crucial point resides in the NC deformation that provides the possibility to join the parameter of noncommutativity, $\theta$, with the inverse of the magnetic field, $B$. On the basis of this relation, and also on the use of the NC version of the Miura transformation which rests on the idea of mapping between the Burgers and $sl_n$-KdV systems, we will show
explicitly how a strong analogy emerges between the Zeeman effect and the consequences of this mapping.

We guess that the incorporation of the Zeeman effect in this context is not a coincidence, we think that some important physical properties are behind it. The starting steps in planting this idea comes from several observations of the behavior of different expressions and also from the primordial role of the NC Burgers system and of the \( \theta \) parameter whose weight increases proportionally with the order of the \( sl_n \)-KdV hierarchy.

We are going to illustrate these ideas for two particular examples, namely the NC KdV and the Boussinesq systems.

1. The \( sl_2 \) KdV- Burgers case

Let us take again the NC KdV-Burgers mapping discussed before, we have the following equation

\[
L_{KdV}(u_2) = p^2 + u_2 = (p + u_1) \star (p - u_1),
\]

or, equivalently, \( u_2 = -u_1^2 - 2\theta u_1' \).

Whereas the NC KdV current \( u_2 \) depends explicitly on the parameter \( \theta \), the Burgers current does not have this property, since the associated Lax operator \( L_{Burgers}(u_1) \) does not admit a Lax operator whose conformal weight is integer as a root. This is also due to the fact that the quantities of non-integer conformal weights are not authorized in this framework.

Thus, from now on, the NC Burgers current is regarded as being a fundamental current in terms of which all the other currents of the \( sl_n \)-KdV hierarchy are expressed.

**Proposition 1:**

Given the NC Miura Transformation, binding the NC Burgers and KdV systems as follows:

\[
L_{KdV}(u_2) = L_{Burg}(u_1) \star L_{Burg}(-u_1),
\]

we can represent this mapping graphically as given by Fig. 1.

**Convention notations:**

For this purpose, we adopt the following diagrammatic representation:

1. We represent symbolically the NC KdV Lax operator by a line indexed by the NC KdV current \( u_2 \). The splitting of \( L_{KdV} \) with respect to the Moyal star product into a pair of NC Burgers operators is schematized by two parallel lines which leave, starting from a vertex, the initial KdV line. The two parallel lines are considered to be associated to the pair
of NC Burgers operators \((L_{Burg}, L_{Burg})\) which appear on the right hand side of the Eq. (14).

2. The position of the two emitted lines relative to the Burgers operators depends on the sign of the spin 1 Burgers current. The upper line is associated with \(L_{Burg}(+u_1)\) while the lower one, associated to negative sign of the NC Burgers current \(-u_1\), is for \(L_{Burg}(-u_1)\).

3. The initial NC KdV line is thus the result of the star product of the parallel Burgers lines.

4. We then specify two zones: the left one, characterized by the NC KdV initial line where the two Burgers levels are in coincidence (degenerated twice). The other zone on the right is given by two separated lines describing a broken degeneracy. The passage from the single KdV level to both Burgers lines, via the Miura transformation or the \(sl_2\) KdV-Burgers mapping, is identical to a lifting of the degeneracy, which means also the passage from a configuration with star product to a configuration without star product:

\[
L_{KdV}(u_2) \hookrightarrow (L_{Burg}(u_1), L_{Burg}(-u_1)),
\]  

or equivalently

\[
(p + u_1) \star (p - u_1) \hookrightarrow ((p + u_1), (p - u_1)).
\]

5. In other words, the transition from a single KdV level degenerated twice to a pair of two Burgers levels without degeneracy is equivalent to the passage from a phase with a \(\theta \equiv B^{-1}\) predominance to a phase with magnetic \(B \equiv \theta^{-1}\) predominance\(^1\). At this point, we have to underline the striking analogy with the Zeeman effect, since it is the presence of a magnetic field which breaks the degeneracy of the initial KdV level.

\[^1\] At the level of the single NC KdV line, where the degeneracy is of order 2, the \(\theta\) parameter acquires a power inversely to the magnetic field \(\theta^{-1}\), which becomes relevant with emission of the pair of NC Burgers levels.
We showed through this first example that the KdV-Burgers mapping is accompanied by a rupture of degeneracy that exhibits the initial NC KdV level. This lifting of degeneracy is due to the emergence of the magnetic field $B$, corresponding to a weakness of the NC deformation parameter $\theta$ during the transition from the single KdV level to the pair of Burgers levels. This behavior is identical to the Zeeman effect.

2. The $sl_3$ Boussinesq-Burgers case

In a similar way, the $sl_3$ Boussinesq-Burgers mapping deals with the following equation:

$$p^3 + u_2 * p + u_3 = (p + u_1) * (p + v_1) * (p - u_1 - v_1).$$

(17)

**Proposition 2:**

Given the NC Miura transformation binding the NC Burgers and $sl_3$ Boussinesq’s system as follows:

$$L_{Bouss}(u_2, u_3) = L_{Burg}(u_1) * L_{Burg}(v_1) * L_{Burg}(u_1 - v_1).$$

(18)

we can represent this mapping graphically as shown by Fig. 2.

**Convention notations:**
In the same way as for the NC KdV-Burgers mapping, we will adopt the following diagrammatic representations for the \( sl_3 \) Boussinesq-Burgers splitting:

1. We represent symbolically the NC \( sl_3 \) Boussinesq Lax operator by a line indexed by the pair of currents \((u_2, u_3)\). The splitting of \( L_{\text{Bouss}} \) with respect to the star product into a triplet \((L_{\text{Burg}}(u_1), L_{\text{Burg}}(v_1), L_{\text{Burg}}(-u_1 - v_1))\) of NC Burgers operators is schematized by three parallel lines, leaving from a vertex the initial \( sl_3 \) Boussinesq line.

2. The position of the three emitted Burgers lines depends on the sign of the spin 1 Burgers currents. The upper lines are chosen arbitrarily to be associated to \( L_{\text{Burg}}(+u_1) \) and \( L_{\text{Burg}}(+v_1) \), while the lower line is associated to the negative sign of the NC Burgers current \(-u_1 - v_1\), namely to \( L_{\text{Burg}}(-u_1 - v_1) \).

3. As is shown in Eq. (17), the initial NC \( sl_3 \) Boussinesq line is the result of the star product of the three parallel Burgers lines.

4. The initial single line corresponding to the NC \( sl_3 \) Boussinesq’s Lax operator is characterized by a degeneracy of order 3. This is due to the fact that the three NC Burgers levels on the right of the vertex in Fig. 2 coincide at the level of the NC \( sl_3 \) Boussinesq line. When the degeneracy of order three is broken, the three NC Burgers levels are emitted. This emission procedure is given by

\[
L_{\text{Bouss}}(u_2, u_3) \leftrightarrow (L_{\text{Burg}}(u_1), L_{\text{Burg}}(v_1), L_{\text{Burg}}(-u_1 - v_1)),
\]

or equivalently

\[
(p + u_1) \star (p + v_1) \star (p - u_1 - v_1) \leftrightarrow ((p + u_1), (p + v_1), (p - u_1 - v_1)).
\]

5. The contact with Zeeman effect is done as follows:

The transition from the initial single NC \( sl_3 \) Boussinesq’s level, of degeneracy three, to a triplet of NC Burgers levels without degeneracy is equivalent to the passage from a phase with a \( \theta \equiv B^{-1} \) predominance to a phase with magnetic \( B \equiv \theta^{-1} \) predominance. At this point, we have to underline the striking analogy with the Zeeman effect, since it is the presence of a magnetic field which breaks the degeneracy of the initial \( sl_3 \) Boussinesq’s level.

We showed once again through this second example that the \( sl_3 \) Boussinesq-Burgers mapping is accompanied by a rupture of degeneracy, of order three, that exhibits the initial NC Boussinesq’s level. This lifting of degeneracy is due to the emergence of the magnetic field \( B \sim \theta^{-1} \) accompanied by an annihilation of the NC deformation parameter \( \theta \) during the emission. This is a clear manifestation of the Zeeman effect.

Before closing this section, we present the Zeeman splitting for the general case of \( sl_n \) KdV-Burgers mapping as given by Fig. 3.
IV. CONCLUDING REMARKS:

This work aims principally to present a new aspect of noncommutative integrable models. We show that the spectrum defined by the Lax pair with spectral parameter for the noncommutative deformation of certain $sl_n$ KdV integrable hierarchies, namely the KdV and Burgers systems, exhibits a degeneracy splitting reminiscent of the Zeeman effect.

The originality of this result and its natural aspect come from the crucial use of the physical property $\theta \sim B^{-1}$, giving the known analogy between the noncommutative parameter $\theta$ and the magnetic field $B$.

The Zeeman effect in the present context seems to be a natural incorporation for the following reasons:

- The $\theta$-Miura transformation is very significant since it is equivalent to a splitting of every $sl_n$ KdV hierarchy into $n$ different Burgers hierarchies.

The associated equation is given by $L_{(KdV)}(u_2) = p + u_2 = (p + u_1) \ast (p - u_1)$.

- The KdV hierarchy described by the NC Lax operator $L_{(KdV)}(u_2)$ and the current $u_2$ of conformal spin-2 such that $u_2 = -u'_1 - 2\theta u'_1$ shows an explicit dependence on the KdV current $u_2 = u_2(\theta)$ in terms of the noncommutative $\theta$ parameter. Unlike the KdV current, the spin-1 Burgers current $u_1$ does not have this property, since the associated NC Lax operator $L_{(Burgers)}(u_1)$ does not admit a Lax operator whose conformal weight is integer...
as a root. This is also due to the fact that the quantities of non-integer conformal weights are not authorized in this framework.

- The NC Burgers current $u_1$ is regarded as being a fundamental current in term of which all the other currents of the $sl\{n\}$-KdV hierarchy are expressed.

- The Miura transformation, applied to $sl\{n\}$-noncommutative hierarchies, shares with the famous Zeeman effect the property of the splitting giving rise in turn to a couple of phases of opposite magnetic field’s dominance.

- In fact, from the Zeeman effect point of view, the NC $sl\{n\}$ KdV hierarchy is associated to a phase $\theta \sim B^{-1}$ where noncommutativity dominates, while the Miura transformation leads to a splitting of the Burgers levels giving rise to a new phase $B \sim \theta^{-1}$ where the magnetic fields dominate.

To make concrete our idea, we have proceeded by presenting a graphical scenario similar to the Zeeman effect representation. The essentials of our representation for the general case, given in Fig. 3, can be summarized as follows:

1. The original NC $sl_n$ Lax operator $L_{sl_n-KdV}$ is represented by a horizontal line indexed by a multiplet of currents $(u_2, u_3, ... u_n)$. These currents indicate that the degree of degeneracy inside this level is of order $n$. Note that the degree of degeneracy is synonymous to the highest degree in the noncommutativity whose parameter is $\theta$.

2. The splitting of $L_{sl_n-KdV}$, with respect to the Miura transformation into a multiplet $(L_{Burg_1}, L_{Burg_2}, ... , L_{Burg_n})$ of NC Burgers operators is schematized by $n$ parallel lines leaving, from a vertex, the initial $sl_n$ KdV line.

3. The position of the $n$ emitted Burgers lines depends on the sign of the spin 1 Burgers currents. Upper lines are for example chosen to be associated to $L_{Burg}(\alpha u_1)$, $\alpha > 0$ while lower lines are associated to the negative sign of the NC Burgers currents, i.e., $L_{Burg}(\alpha u_1)$, $\alpha < 0$.

4. When the degeneracy of order $n$ is broken, the $n$ NC Burgers levels are emitted. This emission procedure is given by

$$L_{Bouss}(u_2, ..., u_n) \hookrightarrow (L_{Burg}(u_1), ..., L_{Burg}(u_n)),$$

or equivalently

$$(p + u_1) \star ... \star (p + u_n) \hookrightarrow ((p + u_1), ..., (p + u_n)).$$

References


