The Berry Phase and the Yang Monopole

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We start with a time-varying $SO(5)$ symmetric matrix and use the Berry phase technique to calculate the vector potential of the Yang monopole which is a generalisation of the Dirac magnetic monopole concept. The calculations are done through simple and straightforward algebraic manipulations.

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I. INTRODUCTION

In a seminal paper [1] Dirac introduced the concept of a non-integrable phase factor in quantum mechanics, and thus initiated the theoretical study of magnetic monopoles. Yang [2] made use of the non-integrable phase factor and generalised the Dirac monopole to an $SU(2)$ monopole in a five-dimensional flat space or four-dimensional spherical space. The Yang monopole possesses $SO(5)$ symmetry. Later Berry [3] proposed that for a time varying hamiltonian the wavefunction would acquire an extra phase now known as the Berry phase. This phase is geometrical in origin. It was demonstrated [4] that the Dirac monopole vector potential could be obtained from a spherical symmetrical and time varying hamiltonian.

In this article we follow the same Berry phase technique to calculate the vector potential of the Yang monopole. At the outset we start with an $SO(5)$ invariant time varying hamiltonian. From the degenerate eigenvectors we can obtain the vector potential of the $SU(2)$ monopole. We have also calculated the vector potential for another set of degenerate eigenvectors. This solution is gauge equivalent to the original Yang solution. The eigenvectors can also be viewed as a rotated state of a fixed spin eigenstate. This is just a generalisation of the spin coherent state. We present a discussion of this via the use of the quaternionic rotational matrix.

This paper is organised as follows. In Sec. II we review the basic procedure of calculating the vector potential of the Dirac monopole using the Berry phase concept. In Sec. III we proceed to calculate the vector potential of the Yang monopole using the same procedure. In Sec. IV we attempt to interpret the eigenvectors of the hamiltonian as a rotated state of the quaternionic rotation matrix. In Sec. V we use the method of contraction to construct an $E(4)$ invariant vector potential in $R^4$. In the last section we present some remarks and a discussion.
II. BERRY PHASE AND THE MAGNETIC MONOPOLE

We examine here a classical example exhibiting the calculation procedure of the magnetic monopole vector potential $A_\mu$ through the Berry phase technique [3, 4]. Consider the time-dependent Hamiltonian

$$H(t) = \sigma \cdot x,$$

where $x$ is a unit vector depending on time $t$ and $\sigma$ are the Pauli matrices. Explicitly $\sigma \cdot x$ is a $2 \times 2$ matrix

$$\sigma \cdot x = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix}.$$  

We have the time evolution equation

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H(t) |\psi(t)\rangle.$$  

First, we have to examine the instantaneous eigenvalue equation

$$\begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix} |n(t)\rangle = \lambda |n(t)\rangle.$$  

It is easy to show that the eigenvalues of this problem are $\pm 1$ for any $t$ and the corresponding eigenvectors can be easily written down. Explicitly for $\lambda = +1$, the corresponding normalised eigenvector is

$$n_+ = \begin{pmatrix} \sqrt{\frac{1 + x_3}{2}} \\ x_1 + ix_2 \\ \sqrt{2(1 + x_3)} \end{pmatrix}.$$  

The Berry phase is obtained through the formula

$$\gamma(T) = i \oint (n(t)|\dot{n}(t)\rangle dt.$$  

In this case we can write the vector potential as

$$A \cdot ds = -i \langle n(t)|dn(t)\rangle,$$

where $A$ is the vector potential for the magnetic monopole field [1] and we have set the electric charge $e$ and magnetic charge $g$ to unity. For $n_+$ we then have the vector potential 1-form $A = A \cdot ds$ as follows,

$$A = \frac{x_1 dx_2 - x_2 dx_1}{2(1 + x_3)}.$$
We can employ spherical coordinates \((x_1, x_2, x_3) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)\) so that
\[
n_+ = \begin{pmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} e^{i\phi} \end{pmatrix}
\]
and
\[
A = \sin^2 \frac{\theta}{2} d\phi.
\]

To understand that this vector potential one form \(A\) would be the solution for the magnetic monopole we can calculate directly the field strength two form \(F = dA\). The result is just the usual monopole magnetic field, which is \(SO(3)\) symmetric and singular at the origin. We would like to make the remark that in our construction of the solution \(SO(3)\) symmetry is maintained throughout. The field strength two form \(F\) transforms invariantly under a gauge transformation. However, the vector potential one form \(A\) transforms inhomogeneously under a gauge transformation, that is, the vector potential one form \(A\) is not explicitly spherically symmetrical. It is spherically symmetrical modulo a gauge transformation \([5]\). Geometrically, the field strength two form \(F\) is just the area two form for the unit sphere \(S^2\). Its integral gives the total magnetic flux which is independent of any distortion of the surface provided that it always encloses the origin.

For the sake of completeness we write down here the normalised eigenvector \(n_-\) for \(\lambda = -1\)
\[
n_- = \begin{pmatrix} -x_1 - ix_2 \\ \sqrt{2(1 + x_3)} \\ \sqrt{1 + x_3} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\sin \frac{\theta}{2} e^{-i\phi} \\ \cos \frac{\theta}{2} \end{pmatrix}
\]
which can give the charge-conjugate solution.

At this point we would like to make some comments. During the time evolution the eigenvalues remain unchanged. Our example is just an isospectral flow problem and we can find the Lax pair representation. For the vector potential let us note that there is a string singularity along the \(z\)-axis. The gauge degree of freedom is a result of the Hopf mapping \([6]\) of \(S^3 \hookrightarrow S^2\). In the next section we shall discuss the next Hopf mapping of \(S^7 \hookrightarrow S^4\) which gives the \(SU(2)\) Yang monopole \([2]\).

**III. THE BERRY PHASE AND THE YANG MONOPOLE**

It is time to make a generalisation. The main obstacle can be overcome if we discard the idea that the \(\sigma\)'s in the previous section are the angular momentum operators. Rather we would recognise that the \(\sigma\)'s form a Clifford algebra, and it is not hard to make the following generalisation. The generalisation is to consider the \(4 \times 4\) matrix
\[
H(t) = \gamma \cdot \mathbf{x},
\]
where \( \mathbf{x} \) is a 5-dimensional unit vector and the \( \gamma \)'s are given by

\[
\gamma_i = \sigma_1 \otimes \sigma_i, \quad i = 1, 2, 3,
\]
\[
\gamma_4 = \sigma_2 \otimes I,
\]
\[
\gamma_5 = \sigma_3 \otimes I,
\]
and they form a Clifford algebra. It is easy to see that

\[
\gamma_i = -\varepsilon_{ijklm} \gamma_j \gamma_k \gamma_l \gamma_m,
\]
where \( \varepsilon \) is the 5-dimensional totally antisymmetric tensor. Explicitly we have

\[
\gamma \cdot \mathbf{x} = \begin{pmatrix}
  x_5 \\
  \sigma_1 x_1 + \sigma_2 x_2 + \sigma_3 x_3 + ix_4 \\
  -x_5
\end{pmatrix}.
\]

The matrix should be considered to be a unitary reflection operator \[7\]. Zhang and Hu \[8\] first considered the same problem by means of spinor coordinates and obtained identical results.

We have two eigenvectors with \( \lambda = +1 \) and two eigenvectors with \( \lambda = -1 \). This degeneracy accounts for the isospin symmetry of \( SU(2) \) \[9\]. We now employ the coordinates

\[
x_5 = \cos \theta_1, \quad x_4 = \sin \theta_1 \frac{1 - \xi^2}{1 + \xi^2}, \quad x_i = \sin \theta_1 \frac{2 \xi_i}{1 + \xi^2}, \quad i = 1, 2, 3.
\]

The \( (\xi_1, \xi_2, \xi_3) \) variables, which parameterise an \( S^3 \) sphere, are a generalisation of the azimuthal circle of the previous problem. In this section we shall use the notation \( e^{i \phi} = \frac{2 \xi}{1 + \xi^2} + i \frac{1 - \xi^2}{1 + \xi^2} \). We can choose the two normalised orthogonal eigenvectors for \( \lambda = +1 \) as

\[
n_+^{(1)} = \begin{pmatrix}
  \cos \frac{\theta_1}{2} \\
  \sin \frac{\theta_1}{2} \frac{2 \xi_3 + i (1 - \xi^2)}{1 + \xi^2} \\
  \sin \frac{\theta_1}{2} \frac{2 (\xi_1 - i \xi_2)}{1 + \xi^2}
\end{pmatrix}
\quad \text{and} \quad
n_+^{(2)} = \begin{pmatrix}
  0 \\
  \cos \frac{\theta_1}{2} \frac{2 (\xi_1 - i \xi_2)}{1 + \xi^2} \\
  \sin \frac{\theta_1}{2} \frac{-2 \xi_3 + i (1 - \xi^2)}{1 + \xi^2}
\end{pmatrix}.
\]

We can now calculate the vector potential 1-form

\[
A_{11} = -i \langle n_+^{(1)} | dn_+^{(1)} \rangle 
= \frac{4 \sin^2 \frac{\theta_1}{2}}{(1 + \xi^2)^2} \left\{ \xi_1 d\xi_2 - \xi_2 d\xi_1 - \frac{1}{2} \left( (1 - \xi^2) d\xi_3 + \xi_3 d\xi^2 \right) \right\},
\]

as well as

\[
A_{12} = -i \langle n_+^{(1)} | dn_+^{(2)} \rangle 
= \frac{4 \sin^2 \frac{\theta_1}{2}}{(1 + \xi^2)^2} \left\{ \xi_2 d\xi_3 - \xi_3 d\xi_2 - \frac{1}{2} \left( (1 - \xi^2) d\xi_1 + \xi_1 d\xi^2 \right) \right\}.
\]
\[-i \frac{4 \sin^2 \frac{\theta_1}{2}}{(1 + \xi^2)^2} \left\{ \xi_3 d\xi_1 - \xi_1 d\xi_3 - \frac{1}{2} \left[(1 - \xi^2) d\xi_2 + \xi_2 d\xi^2 \right] \right\}.
\]

Defining the gauge potential $b_{i\mu}$ as

\[
b_{i\mu} dx^\mu = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},
\]

(19)

with the generators $X_i$, $i = 1, 2, 3$ of the gauge group being identified as $\sigma_i/2$, respectively, we thus get the $\alpha$ solution of the Yang monopole [2]. We can get the $\beta$ solution by the use of the $\lambda = -1$ eigenvectors.

As discussed in [2] this vector potential one form $b$ gives an $SO(5)$ symmetrical field strength two form $f = db + b \wedge b$, which transforms covariantly under a gauge transformation and is singular at the origin. The four form $f \wedge f$ can be integrated giving rise to the hypersurface area of the unit sphere $S^4$, which is the second Chern class. This is a generalisation of our abelian case of the total flux enclosed, which is the first Chern class.

The arbitrariness in choosing the two degenerate orthogonal eigenvectors is the gauge freedom. Indeed in another way, we can choose the two eigenvectors of $\lambda = +1$ as

\[
n^+ = \begin{pmatrix} \cos \frac{\theta_1}{2} n^+ \\ \sin \frac{\theta_1}{2} e^{i\phi} n^+ \end{pmatrix},
\]

(20)

\[
n^- = \begin{pmatrix} \cos \frac{\theta_1}{2} n^- \\ -\sin \frac{\theta_1}{2} e^{-i\phi} n^- \end{pmatrix},
\]

(21)

where $n^+$ and $n^-$ are the eigenvectors of our previous section. We would like to call this the chiral gauge. We can now calculate

\[
A_{++} = -i (n^+_+ | dn^+_+ ) = \sin^2 \frac{\theta_1}{2} d\phi - i (n^+_+ | dn^+_+ ) .
\]

(22)

With our $\xi$ parametrisation we can write

\[
A_{++} = -\sin^2 \frac{\theta_1}{2} \frac{2d\xi}{1 + \xi^2} + \frac{\xi_1 d\xi_2 - \xi_2 d\xi_1}{2(\xi + \xi_3)} .
\]

(23)

We can also calculate

\[
A_{++} = -i (n^+_+ | dn^-_+ ) = -i \left( \cos^2 \frac{\theta_1}{2} - \sin^2 \frac{\theta_1}{2} e^{-i2\phi} \right) (n^+_+ | dn^-_+ ) .
\]

(24)

With the $\xi$ parametrisation, we have

\[
A_{++} = -i \left( \cos^2 \frac{\theta_1}{2} - \sin^2 \frac{\theta_1}{2} e^{-i2\phi} \right) \left\{ \frac{(\xi_1 d\xi_3 - \xi_3 d\xi_2) - i(\xi_2 d\xi_3 - \xi_3 d\xi_2)}{2\xi(\xi + \xi_3)} - \frac{d\xi_1 - i d\xi_2}{\xi + \xi_3} + \frac{(\xi_1 - i\xi_2) d\xi}{\xi(\xi + \xi_3)} \right\} .
\]

(25)
We can then derive the vector potential $b$, but the expression seems complicated. We can employ another co-ordinate system

\begin{align*}
  x_1 &= \sin \theta_1 \sin \theta_2 \sin \theta_3 \sin \theta_4, \\
  x_2 &= \sin \theta_1 \sin \theta_2 \sin \theta_3 \cos \theta_4, \\
  x_3 &= \sin \theta_1 \sin \theta_2 \cos \theta_3, \\
  x_4 &= \sin \theta_1 \cos \theta_2, \\
  x_5 &= \cos \theta_1.
\end{align*}

Then we have

\begin{align*}
  A^{\pm \pm}_{++} &= \sin^2 \frac{\theta_1}{2} d\theta_2 + \sin^2 \frac{\theta_3}{2} d\theta_4, \tag{27}
  \\
  A^{\pm +}_{++} &= \left( \cos^2 \frac{\theta_1}{2} - \sin^2 \frac{\theta_3}{2} e^{-i\theta_2} \right) \left( \frac{1}{2} \sin \theta_3 d\theta_4 + \frac{1}{2} i d\theta_3 \right) e^{-i\theta_4}. \tag{28}
\end{align*}

Of course this solution is gauge equivalent to our original Yang solution.

**IV. QUATERNIONIC WAVEFUNCTION**

As we have mentioned previously, the Dirac monopole is related to the Hopf mapping $S^3 \mapsto S^2$, and the Yang monopole is related to the Hopf mapping $S^7 \mapsto S^4$. This involves the generalisation of the division algebra of complex numbers to quaternions, which are non-commutative. The discussion of the connection of quaternions with the $SU(2)$ monopole was first put forth by Minami [10]. Previously we [11, 12] have proposed that in the Dirac monopole case the eigenvector can be regarded as a rotated state of the spin eigenstate. In the mathematical parlance this is just the spin coherent state [13]. So here we employ this idea to get the rotated state via the rotation matrix with quaternionic entries. We then proceed to derive the Yang $SU(2)$ monopole solution.

First of all we have to replace the entries in the rotation matrix by quaternionic variables $p$ and $q$. Any quaternionic variable $q$ can be written as

\begin{align*}
  q &= q_0 + q_1 e_1 + q_2 e_2 + q_3 e_1 e_2, \\
  &= (q_0 + q_1 e_1) + (q_2 + q_3 e_1) e_2. \tag{29}
\end{align*}

where $q_0$, $q_1$, $q_2$, and $q_3$ are real variables and $(e_1, e_2)$ are subjected to the algebra

\begin{align*}
  e_1^2 &= e_2^2 = -1, \tag{30}
  \\
  e_1 e_2 &= -e_2 e_1. \tag{31}
\end{align*}
A generalised rotation of the quaternionic states is effected by a \(2 \times 2\) matrix with quaternion entries,
\[
U = \begin{pmatrix} p & -\bar{q} \\ q & \bar{p} \end{pmatrix},
\]
(32)
with
\[
\bar{p}p + \bar{q}q = 1.
\]
(33)
This defines a topological space \(S^7\). In analogy we can define a specific rotation of \(S^4\) such that
\[
\begin{pmatrix} \cos \frac{\theta}{2} & -\sin \frac{\theta}{2} \bar{r} \\ \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix},
\]
(34)
where \(r\) is a unit quaternion variable. Following Yang [2] we introduce the \((\xi_1, \xi_2, \xi_3)\) variables which parameterise the \(S^3\) sphere. The unit quaternion \(r\) can be written as
\[
r = \frac{1 - \xi^2}{1 + \xi^2} + \frac{2\xi_1 e_1 + 2\xi_2 e_2 + 2\xi_3 e_3}{1 + \xi^2},
\]
(35)
where \(\xi^2 = \xi_1^2 + \xi_2^2 + \xi_3^2\). The Hopf mapping \(S^7 \rightarrow S^4\) allows for the degrees of freedom of an \(SU(2)\) gauge field. The rotated quaternionic states are
\[
u' = u \cos \frac{\theta}{2} + v \sin \frac{\theta}{2} \bar{r},
\]
(36)
\[
u' = -u \sin \frac{\theta}{2} \bar{r} + v \cos \frac{\theta}{2},
\]
(37)
where \(u\) and \(v\) are the basic up and down eigenstates. The inner product
\[
(u', du')
\]
(38)
defines a pure imaginary quaternion one-form. To identify this with the \(SU(2)\) monopole field we have the mapping
\[
(e_1, e_2, e_1e_2) \rightarrow -i(\sigma_1, \sigma_2, \sigma_3).
\]
(39)
so that
\[
(u', du') \rightarrow b^1_\mu X_\mu dx^\mu,
\]
(40)
where we have taken \(\sigma_j/2\) for \(X_j\), the generators of the \(SU(2)\) gauge group. Explicitly, we have
\[
b^1_\mu dx^\mu = \frac{8}{(1 + \xi^2)^2} \left\{ \xi_3 d\xi_2 - \xi_2 d\xi_3 + \frac{1}{2} ((1 - \xi^2) d\xi_1 + \xi_1 d\bar{\xi}^2) \right\}
\]
(41)
and cyclic permutations. This is just the \(\alpha\) solution in Yang [2]. To get the \(\beta\) solution we can proceed with the rotated state \(v'\).
V. METHOD OF CONTRACTION

It is well-known that we can use the method of contraction to get an $E(2)$ invariant magnetic field from the $SO(3)$ Dirac monopole field. We can take the approximation $x_3 \rightarrow 1$ and $x_1$ and $x_2$ infinitesimal small in the exact wavefunction. That is to say we consider the flat portion of the north pole. The wavefunction becomes

$$n_+ = \left( \frac{1}{x_1 + ix_2} \right). \tag{42}$$

This formula can also be obtained by a perturbation calculation of the eigenvalue problem of the matrix equation. So the vector potential $A$ is

$$A = \frac{x_1 dx_2 - x_2 dx_1}{4}, \tag{43}$$

which is just the vector potential for a constant magnetic field in the symmetric gauge. We have the symmetric constraint $x_1^2 + x_2^2 = \text{constant}.$

Analogously we can obtained an $E(4)$ invariant gauge field by the method of contraction. The contracted wavefunctions will be

$$n_+^{(1)} = \begin{pmatrix} 1 \\ 0 \\ \frac{x_3 + ix_4}{2} \\ \frac{x_1 + ix_2}{2} \end{pmatrix}, \quad n_+^{(2)} = \begin{pmatrix} 0 \\ 1 \\ \frac{x_1 - ix_2}{2} \\ -\frac{x_3 + ix_4}{2} \end{pmatrix}. \tag{44}$$

The vector potential $A$ of the gauge field will be

$$A_+^{++} = \frac{x_1 dx_2 - x_2 dx_1 + x_3 dx_4 - x_4 dx_3}{4}, \tag{45}$$

$$A_+^{+-} = \frac{(x_2 dx_3 - x_3 dx_2 + x_1 dx_4 - x_4 dx_1) - i(x_3 dx_1 - x_1 dx_3 + x_2 dx_4 - x_4 dx_2)}{4}. \tag{46}$$

VI. CONCLUSION

In this paper we have offered a more algebraic and layman method to calculate the vector potential of the Yang monopole. The original calculation of Yang proceeded with the non-integrable phase factor and imposed the spherically symmetric condition. In our approach we start with a spherically symmetrical hamiltonian to get the Berry phase. We then extract the vector potential from the Berry phase. Alternatively, we can introduce a quaternionic wavefunction of a spin coherent state and calculate the vector potential. The monopole harmonics [14] of the Yang monopole have found application in the four-dimensional quantum hall effect [8]. We hope that our way of calculation can make things
simpler. This solution can be regarded as the starting point of maximal symmetry. By the method of contraction we can obtain an $E(4)$ invariant vector potential on $R^4$. It is conceivable that we can restrict the symmetry and obtain a vector potential with lesser symmetry on a lower dimensional space.

References