The KMS State of Spacetime at the Planck Scale

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We explore here the consequences of the expected thermal equilibrium of the spacetime system at the Planck scale. In the context of string theory, it is supposed that at high temperature (Planck temperature) this system is subject to a $4D \to 3D$ dimensional reduction, the Chern-Simon term becoming dominant in the action. We then show that the spacetime must be considered as being subject to the Kubo-Martin-Schwinger (KMS) condition at the Planck scale. Therefore, in the interior of the KMS strip, i.e., from the scale $\beta = 0$ to the scale $\beta = \ell_{Planck}$, the time-like direction should be viewed as complex, the two real poles being $\beta = 0$ and $\beta = \ell_{Planck}$. This means that, within the limits of the KMS strip, the spacetime metric should be considered as subject to quantum fluctuations between the Lorentzian (physical) state and the Euclidean (topological) state.

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I. Introduction

Much have been recently proposed regarding the physical state of the universe at the vincinity of the Planck scale. String theory and supergravity or, on a more formal basis, non-commutative geometry, have contributed, independently of each others, to establish on a convincing basis the possible exitence of a "transition phase" in the physical content and the topological structures of the (pre)spacetime at very high temperature (i.e., the Planck temperature). But what can reasonably be said concerning such a transition? In the present paper, we consider the interesting consequences of the expected thermodynamical equilibrium state predicted by the Standard Model for the (pre)universe at the Planck scale. This notion of equilibrium state of the spacetime at high temperature has been suggested and investigated a long time ago in the context of standard quantum field theory [1-4]. Recently, it has been reformulated with some new interesting arguments [5-6] and, in a different context, [7-9]. As a natural consequence of this thermal equilibrium, according to [10], we then suggest hereafter that the (pre)universe should be considered as *subject to the Kubo-Martin-Schwinger* (KMS) *condition* [11] at the Planck scale. This point of view is in accordance with the fact that at high temperature (i.e. the Planck scale) the (pre)space-time is generally supposed, in the context of string theory, to be subject to a $4D \to 3D$ dimensional reduction. Such a process has been investigated in the string context by Seiberg and Witten in [12]. Let's remark that in this case, the Chern-Simon term becomes relevant in the action of the residual three-dimensional spacetime, the metric being *independent* of the fourth coordinate [10]. But as demonstrated in [13], such a property is directly related to the KMS state. Now, what are the main physical consequences of this possible KMS state of the spacetime? Indeed, they might be important, because when applied to quantum spacetime, the KMS statistics are such that, within the limits of the "KMS strip" (i.e. between the scale zero and the Planck scale), the time-like direction of the system should be considered as *complex*: $t \mapsto t_c = t_r + it_i$. G. Bogdanoff has showed in [10] that at the scale zero, the theory is projected onto the pure imaginary boundary $tH \mapsto t_c = it_i$ of the KMS strip. Namely, there exists, around $\beta \to 0$, a non-trivial *topological* limit of quantum field theory, *dual* to the usual topological limit associated with $\beta \to \infty$ in the partition function (2). Such a topological state of the (pre)spacetime has been detailed in [13].

The present article is organized as follows. In section 2 we recall that at the Planck scale, the "spacetime system" should most likely be considered as subject to a 4D $\to$ 3D dimensional reduction. Interstingly, such a reduction could appear as a condition of the KMS condition. In section 3, we consider the expected thermodynamical equilibrium state of the system. In section 4, we show that, as a consequence of this equilibrium state, the spacetime must be considered as subject to the KMS condition. In section 5, we suggest that, as a result of the KMS properties, the time-like direction g_{44} of the metric should be seen as complex $t_c = t_r + it_i$ within the limits of the KMS strip. In section 6, we discuss the transition from imaginary time t_i (scale zero) to real time t_r (Planck scale) in terms of KMS breaking beyond the Planck scale.

II. Preliminaries: 3-dimensional topological theory at the planck scale

It is currently admitted in quantum field theory as in the superstrings context that at high temperature (here the Planck temperature) the 4-dimensional Lorentzian theory is subject to a dimensional *reduction* onto a 3D theory [12]. So the theory is defined by three-manifold invariants (in particular the Floer invariant of a supersymmetric non linear σ-model [13]). Surprisingly, this has important consequences on the possible KMS state of the spacetime at the Planck scale. Indeed, it is intersting to observe that, as a natural result of the 4D $\to$ 3D dimensional reduction, the three dimensional pseudo-gravity $\Gamma_{(3)}$ is then coupled to the **S**, **T** complex scalar fields:

- the dilaton field $S = \dfrac{1}{g^2} \pm i.\alpha_-$ (axion) with **S** and S^*

- the **T**-field $T = g_{44} \pm i g^*_{-4}$ with **T** and T^*

This is a well known result in string theory. An interesting aspect of the two hereabove propagating scalar fields is that they are exhibiting two important dualities:

(i) the **S**-duality, whose group is SL(2, $\mathbb{Z}$). Any vector (F_D, F) is transformed into the following doublet:

$$S : \begin{pmatrix} F_D \\ F \end{pmatrix} \xrightarrow{S} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} F_D \\ F \end{pmatrix} .$$

(ii) the **T**-duality, whose group is O(p, q; $\mathbb{Z}$) and which gives the transformation:

$$T : \begin{pmatrix} F_D \\ F \end{pmatrix} \xrightarrow{T} \begin{pmatrix} F_D + F \\ F \end{pmatrix} .$$

The complex dilaton **S**-field $g \to \frac{1}{g}$ exchanges strong coupling of a theory and weak couplings of the dual theory. Likewise, the **T**-field, $R \to \frac{1}{R}$ exchanges large and small radius. Now, we claim that these spin 0 fields, coupled to the metric tensor field, can be seen as a source of "dualisation" between the Lorentzian quantum field theory at the Planck scale and the Euclidean topological theory at zero scale. So we suggest a new type of duality, (which we call "i-duality" $t \to \frac{1}{it}$ [10]), which exchanges real time in strong coupling / large radius and imaginary time in weak coupling / small radius. In this sense, Planck (physical) scale is i-dual to zero (topological) scale. In such a context, the 3-dimensional theory is *independent* of the fourth coordinate g_{44}. Then, the 3D "action" is topological, given by the well known Chern-Simon term [14]:

$$S_{Chern-Simon} = Tr(\omega \wedge d\omega + \frac{2}{3}\omega \wedge \omega \wedge \omega). \tag{1}$$

This is very important because as proved in [10-13], the fact that the Chern-Simon topological term becomes relevant for the three dimensional space-Iike surface implies in a quite natural way that the four dimensional spacetime can be seen undifferently as Lorentzian or Euclidean. Indeed, the coupling of the **S**/**T**-fields and the 3D pseudo-gravity is given by the σ-model:

$$\sum = SO(3) \times \frac{SL(2,R)}{SO(2)} \times \frac{SL(2,R)}{SO(1,1)}. \tag{2}$$

As the 3D theory is *independent* of g_{44}, the 2D field $\frac{SL(2,R)}{SO(1,1)}$ in the Lorentzlan case and $\frac{SL(2,R)}{SO(2)}$ in the Euclidean case can be viewed as equivalent. The form of the 5D metric is [13]:

$$ds^2 = a(w)^2 d\sigma^2_{(3)} + \frac{dw^2}{g^2} - dt^2, \tag{3}$$

where the axion term is $a = f(w,t)$, the 3-geometry being $d\sigma^2_{(3)} = f(x,y,z)$. Clearly, the expected values of the running coupling constant $\alpha = \frac{1}{g^2}$ are giving the two 4D limits of the 5D metric of Equ. (3). Therefore, as showed in [13], this 5D configuration can be seen as a quantum "superposition state" of the two 4D metrics and corresponds to the KMS state of the (pre)spacetime at the Planck scale. At the Lie groups level, the corresponding "superposition state" of the signature $(+ + + \pm)$ can be described by the symmetric homogenous space

$$\sum_h = \frac{SO(3,1) \bigoplus SO(4)}{SO(3)},$$

SO(3) being diagonally embedded in SO(3, 1) $\bigoplus$ SO(4).

In conclusion, the relevance of the Chern-Simon term at high temperature (Planck scale) suggests that it is reasonable to envisage that the spacetime could be subject to the KMS condition at such a scale. Let's go now further in this direction in considering the consequences of the thermodynamical equilibrium of the spacetime at the quantum scale.

III. Thermodynamical equilibrium of the spacetime at the planck scale

It is well known that at the Planck scale, one must expect a thermodynamical *equilibrium state* characterizing, most likely, the (pre)spacetime at such a scale. To begin with, the seminal investigations of K. Huang and S. Weinberg [2], L. Dolan and R. lackiw [3] have renewed the initial idea of Hagedorn concerning the existence, at very high temperature, of a limit restricting the growth of states excitation. In this perspective, J. J. Atick and E. Witten have shown the existence of a Hagedorn limit around the Planck scale in string theory [1]. As recently recalled by C. Kounnas [6] in the context of $N = 4$ superstrings, the reason of such a limit is that at finite temperature, the partition function $Z(\beta)$ and the mean energy $U(\beta)$ develop some power pole singularities in $\beta \equiv T^{-1}$ since the density of states of a system grows exponentially with the energy E:

$$\begin{aligned} Z(\beta) &= \int dE\,\rho(E)\,e^{-\beta E} \approx \frac{1}{(\beta-\beta_b)^{(k-1)}} \\ U(\beta) &= -\frac{\partial}{\partial\beta}\ln Z \approx (k-1)\frac{1}{\beta-\beta_b} \end{aligned} \tag{4}$$

Obviously, one can infer from Eq. (4) the existence, around the Planck scale, of a critical temperature $T_H = \beta_b^{-1}$, where the (pre)spacetime system must be viewed as in a thermodynamical equilibrium state. Indeed, $a(t)$ being the cosmological scale factor, the global temperature T follows the well-known law:

$$T(t) = T_p\frac{a(t_p)}{a(t)}$$

and around the Planck time, T is reaching the critical limit $T_p \simeq \frac{E_p}{k_B} \simeq \left(\frac{\hbar C^5}{G}\right)^{\frac{1}{2}} k_B^{-1} \simeq 1,4.10^{32}$ K. As far as the ratio between the interaction rate (Γ) of the initial fields and the (pre)spacetime expansion (H) is currently admitted to be $\frac{\Gamma}{H} \ll 1$, the system can reasonably be considered in *equilibrium state*. This has been convincingly established a long time ago within some precursor works [1, 2, 3] and more recently in quantum field theory [4] and in the superstrings context [5]. However, this natural notion of equilibrium, when viewed as a global gauge condition, has dramatic consequences regarding physics at the Planck scale. Among those consequences, the most important is that the (pre)spacetime at the Planck scale should be considered as subject to the famous "KMS condition", a very special and interesting physical state that we are now going to investigate.

IV. The (pre)spacetime in KMS state at the planck scale

The KMS theory has been succesfully constructed some years ago in a few fundamental papers [11]. As recalled in [10] , it appears that the notion of KMS state is deeply related to the equilibrium state of a system. Let's first recall on mathematical basis what an equilibrium state is.

Definition *H being an autoadjoint operator and* K *the Hilbert space of a finite system, the equilibrium state* ω *of this system is described by the Gibbs condition* $\omega(A) = \frac{Tr_K(e^{-\beta H}A)}{Tr_K(e^{-\beta H})}$ *and satisfy the KMS condition.*

This well-known definition has been notably proposed in [11]. Now, it is natural to oppose the notion of equilibrium to the one of *evolution* of a system. With this in view, the famous Tomita-Takesaki modular theory has established that the dynamic of a quantum system corresponds, in a unique manner, to the strongly continuous one parameter *-automorphism group α_t of some von Neumann C*-algebra **A** [15]:

$$\alpha_t(A) = e^{iHt} A e^{-iHt}. \tag{5}$$

This one parameter automorphisms group describes the time evolution of the observables of the system. At this stage, we are brought to find the remarkable discovery of Takesaki and Winnink, which relates the evolution group $\alpha_t(A)$ of a system (i.e. the modular group $M = \Delta^{it} M \Delta^{-it}$) with the equilibrium state $\varphi(A) = \frac{\mathrm{Tr}(Ae^{-\beta H})}{\mathrm{Tr}(e^{-\beta H})}$ of this system [11-16]. This deep relation between evolution $\alpha_t(A)$ and equilibrium $\varphi(A)$ characterises the famous "KMS condition".

Let's recall now how such a relation between equilibrium state and evolution of a system is realized by the KMS condition. It has been clearly established [11] that a state ω on the C*-algebra A and the continuous one parameter automorphism group of A at the temperature $\beta = 1/kT$ verify the KMS condition if, for any pair A, B of the *-sub-algebra of A, α_t-invariant and of dense norm, it exists a $f(t_c)$ function holomorphic in the strip $\{t_c = t + i\beta \in C$, Im$t_c \in [0, \beta]\}$ such that:

$$\begin{aligned} &(i) \quad f(t) = \varphi(A(\alpha_t B)); \\ &(ii) \quad f(t + i\beta) = \varphi(\alpha_t(B)A); \quad \forall t \in R. \end{aligned} \tag{6}$$

The above definition expresses the bijective relation between equilibrium state, holomorphic state of the measure parameters and KMS state.

Now, considering the general properties raised by the KMS condition, if we admit that around the Planck scale, the (pre)spacetime system is in a thermal equilibrium state, then we are also bound to admit that this system is in a KMS state. Indeed, it has been shown a long time ago [11] that if a state of a system ω satisfies the equilibrium condition $\int_{-\infty}^{\infty} \omega([h, \alpha_t(A)])dt = 0$, $\forall A \in U$ then, ω satisfies the KMS condition. So, there is a biunivoque relation between equilibrium state and KMS state. So, if we admit that around ℓ_{Planck}, the (pre)spacetime system is in a thermal equilibrium state, then according to [11], we are also bound to admit that this system is in a KMS state.

Next, let's push forwards the consequences raised by the holomorphicity of the KMS strip.

V. Holomorphic time flow at the planck scale

A determinant (and unexpected) consequence of the application of the KMS condition to the spacetime concerns the spacetime metric itself. As a direct effect of the KMS properties, the time-like coordinate g_{00} should, in a natural manner, be considered as holomorphic within the limits of the KMS strip. As demonstrated in details in [10-13], within the holomorphic KMS strip, we necessarily should have:

$$t \to t_c = t_r + it_i. \tag{7}$$

This point of view has been implicitely adopted by Derredinger and Lucchesi in [17]. For these authors, the Boltzmann weight $e^{-\beta H}$ can be seen as an evolution operator in Euclidean time, so that, after a translation in imaginary time, we get the formula (8):

$$e^{i\beta H}\phi(t;x)e^{\beta H} = \phi(t + i\beta;x) \tag{8}$$

which is naturally inducing the notion of complex time like direction. In the same way, the physical (real) temperature should also be considered as complex at the Planck scale:

$$T \to T_c = T_r + iT_i \tag{9}$$

as proposed by Atick and Witten in another context [1]. This interesting effect is simply due to the fact that, given a von Neumann algebra W^* and two elements A, B of W^*, then there exists a function $f(t_c)$ holomorphic in the strip $\{t_c \in \mathbb{C}, \mathrm{im}\,t_c \in [0;\hbar\beta]\}$ such that:

$$f(t) = \varphi(A(\alpha_t B)) \text{ and } f(t_r + i\hbar\beta) = \varphi(\alpha_t(B)A); \forall t \in \mathbb{R} \tag{10}$$

t being the usual time parameter of the 3D theory, like $\hbar\beta = \hbar/kT$. So in our case, within the limits of the KMS strip, i.e. from the scale zero ($\beta = 0$) to the Planck scale ($\beta = \ell_{Planck}$), the "time-like" direction of the system must be extended to the complex variable

$$t_c = t_r + i\,t_i \in \mathbb{C}; \mathrm{Im}\,t_c \in [i\,t_i;t_r]: \tag{11}$$

Of course, the holomorphicity of the time like direction of the spacetime is induced in a natural manner by the fact that in our approach, the thermodynamical system is the spacetime itself. Such a situation has been investigated on formal basis in [10], notably in the context of "quantum groups" and non-commutative geometry. Interestingly, it has been established by G.F.R. Ellis *et al.* in the relativistic context that the signature change and, more generally, the notion of complex time-like direction are compatible with the constraints raised by the field equations [18]. More recently, the compatibility between relativistic constraints and KMS condition has also been clarified by J. Bros and D.A. Buchholz in [19].

Next step, according to Tomita's modular theory [16-20], the KMS condition, when applied to the spacetime as a global system, allows, within the KMS strip, the existence of an "extended" (holomorphic) automorphism "group of evolution". Such an extended group depends, in the classification of factors [12], on a "type III$_\lambda$ factor" M_q (a factor is a special type of von Neumann algebra, whose the center is reduced to the scalars $a \in \mathbb{C}$). The "extended" automorphism group has the following form:

$$M_q \mapsto \sigma_{\beta_c}(M_q) = e^{H\beta_c} M_q e^{H\beta_c} \tag{12}$$

with the $\beta_c = \beta_r + i\hbar\beta_i$ parameter being formally *complex*. One can interprete β_c as a complex time t and / or temperature $T \to T_c = T_r + iT_i$. So, the KMS condition suggests the existence at the Planck scale, of an effective one loop potential coupled, in $N = 2$ supergravity, to the complex dilaton + axion field (i.e. the S-field in string theory) $\varphi = \frac{1}{g^2} + i\alpha$ yielding the dynamical form $\eta_{1o} = \mathrm{diag}(1;1;1;e^{i\theta})$ for the metric. As detailed in [10], the coupling of the S/T-fields with the 3D pseudo-gravity is given by the extended σ-model:

$$\Sigma = SO(3) \times \frac{SL(2;\mathbb{R})}{SO(2)} \times \frac{SL(2;\mathbb{R})}{SO(1;1)}: \tag{13}$$

In this case, the "action" of the 3D pseudo-gravity is given by the Chern-Simon term, the theory being *independent* of the fourth direction g_{44} of the metric. The 2D field $\frac{SL(2;R)}{SO(1;1)}$ in the Lorentzian case and $\frac{SL(2;R)}{SO(2)}$ in the euclidean case can be viewed as equivalent. Consequently, the signature of ´₁ₒ is Lorentzian (i.e. physical) for $\mu = §¼$ and is allowed to become Euclidean (topological) for $\mu = 0$. Therefore, the "KMS signature" of the metric can be seen as in a quantum superposition state, of the form $(+ + +§)$. This is as it should be since, considering the quantum fluctuations of g₁ₒ, there is no more invariant measure on the non commutative metric (for details, see [13]). Therefore, according to von Neumann algebra theory, the "good factor" addressing those constraints is uniquely a non commutative *traceless* algebra, i.e. the type III¸ factor M_q, of the general form constructed by Connes [15]:

$$M_q = M^{0;1}_{Top} > /_{\mu} R^{¤}_{+} =^{-} Z \,´\, M^{0;1}_{Top} > /_{\mu}{}^{-} S_1 \quad (14)$$

the "topological" factor $M^{0;1}_{Top}$ being a type II$_1$ factor and $R^{¤}_{+}=^{-}Z$ the group acting periodically on $M^{0;1}_{Top}$. We have shown in [13] that Equ. (14) can be equivalently expressed under the extended form $M_q = M^{0;1}_{Top} > /_{\mu} M_{Phys}$, with M_{Phys} being a *type* I$_1$ factor, indexed by the real group R, which we consider hereafter as a "physical factor" (for details, see Ref. [10]) .The relation between the periods ¸ and ¯ is such that ¸ $= \frac{2¼}{¯}$, so that when ¯ ! 1, we get ¸ ! 0 (the periodicity is suppressed) and we are left with the unique physical factor M_{Phys} in the usual real time evolution context. On the contrary, for ¯ $= 0$, the theory is defined by the topological factor $M^{0;1}_{Top}$ giving the "pseudo-evolution" in imaginary time and becomes purely topological. Again, see Ref. [13] for details.

From another point of view, we can also consider the well known Conne's invariant

$$\pm : R \ ! \ \mathrm{OUT}\, M = \frac{\mathrm{AUT}\, M}{\mathrm{INT}\, M}; \quad (15)$$

OUT M representing the exterior automorphisms of the algebra M, INT M the inner automorphisms (necessarily present in the non commutative case) and AUT M the automorphisms of M (see Ref. [15] for definitions). Now, starting from Eq. (15), we have constructed in [13] the extension *Ext* (noted T) of OUT M_q by INT M_q in AUT M_q:

$$\mathrm{AUT}\ M_q \,´\, \mathrm{OUT}\ M_q\ T\ \mathrm{INT}\ M_q \quad (16)$$

with $fx;yg$ 2 OUTM_q and $fx^0;y^0g$ 2 INTM_q. We have shown that ¾¯$_c(M_q)$ ½ OUT M_q T INT$_{mod}M_q$, so that we observe that AUT M_q is clearly corresponding to $M^{0;1}_{Top} > /_{\mu}{}^{-}S_1$, i.e. the "unification of the two factors $M^{0;1}_{Top}$ and M_{phys} within the unique "quantum" III¸ factor M_q.

At the Lie group level, this "superposition state" can simply be given by the symmetric homogeneous space constructed in [10] :

$$\mathbf{X}_h = \frac{SO(3;1) \,\mathbf{L}\, SO(4)}{SO(3)} \quad (17)$$

to which corresponds, at the level of the underlying metric spaces involved, the topological quotient space:

$$\mathbf{X}_{top} = \frac{R^{3;1} \,\mathbf{L}\, R^4}{SO(3)}: \quad (18)$$

In the non commutative context, G. Bogdanoff has constructed, again in [10], the "cocyle bi-crossproduct":

$$U_q(so(4)^{op\ Ã}\ .\ /\ U_q(so(3;1)) \tag{19}$$

where $U_q(so(4)^{op}$ and $U_q(so(3;1))$ are Hopf algebras (or "quanturn groups" [21]) and Ã a 2-cocycle of q-deformation. The bicrossproduct (19) suggests an unexpected kind of "unification" between the Lorentzian and the Euclidean Hopf algebras at the Planck scale and yields the possibility of a "q-deformation" of the signature from the Lorentzian (physical) mode to the Euclidean (topological) mode [10-13].

Now, let's go back to Eq. (14) in order to precise the notion of "unification" between topological and physical states at the Planck scale. It appears clearly that the Tomita-Takesaki modular automorphisms group $¾_{¯_c}(M_q)$ corresponds to the coupling given by Eq. (14) and induces, within the KMS field, the existence of *two* dual flows.

(i) On the boundary $¯_i\ ¸\ `_{Planck}$, the first possible flow, of the form

$$¾_t(M_q) = e^{iH¯} M_q e^{i\ iH¯} \tag{20}$$

represents the well known algebra of observables of the system and corresponds to the Lorentzian flow in real time. As proposed herabove, this flow can be seen as a "physical flow", that we call $P^f_{¯>0}$. The corresponding scale represents the physical part of the light cone and, consequently, the notion of (Lebesgue) measure is fully defined. Therefore, the (commutative) algebra involved at such a scale is endowed with a hyperfinite trace and is given on the infinite Hilbert space $L(H)$, with $H = L^2(R)$. Then, $L(L^2(R))$ is a *type* I_1 factor, indexed by the real group R, which we have called M_{phys}. At this scale, the theory is Lorentzian, controlled by SO(3, 1).

(ii) On the topological "zero scale" $¯_i = 0$ limit, the second flow takes necessarily the non unitary form:

$$¾_{i¯}(M_q) = e^{¯_r H} M_q e^{i\ ¯_r H} \tag{21}$$

giving on M_q the semi-group of unbounded and non-stellar operators. As proposed in [13], this initial "topological" scale corresponds to the imaginary vertex of the light cone, i.e. a zero-size gravitational instanton [10]. All the measures performed on the Euclidean metric being ½-equivalent up to infinity, the system is ergodic. A. Connes has shown on general basis that any ergodic flow for an invariant measure in the Lebesgue measure class gives a unique type II_1 hyperfinite factor [15]. This strongly suggests that the singular 0-scale should be described by a type II_1 factor, endowed with a hyperfinite trace noted Tr_1. We have called $M^{0;1}_{Top}$ such a "topological" factor, which is an infinite tensor product $©^1$ of matrices algebra (ITPFI) of the $R_{0;1}$ Araki-Woods type [22]. As pointed hereabove, since $M_q = M^{0;1}_{Top} > /_{\mu}\ ¯S_1$, on th $¯_i = 0$ limit, we get $M_q\ ´\ M^{0;1}_{Top}$. With respect to the analytic continuation between (20) and (21), $¾_{i¯}(M_q)$ represents a "current in imaginary time". We have stated in [13] that this current can be seen as another way to interpret the "flow of weights" of the algebra M_q [23]. Clearly, according to [23], the flow of weights of M_q is an ergodic flow, which represents an invariant of M_q. Then, $¾_{i¯}(M_q)$ yields a pure topological amplitude [24] and, as such, "propagates" in imaginary time from zero to infinity. $¾_{i¯}(M_q)$ is not defined on the whole algebra M_q but on an ideal fTg of M_q. One can demonstrate that in this case, the theory is Riemannian, the isometries of the metric being given by SO(4). As showed in [10, 13] this zero scale corresponds to the first Donaldson invariant $I = \sum_i (-1)^{n_i}$ and can be described by the topological quantum field theory proposed by E. Witten in [24].

To finish, let's observe that the topological flow does not commute with the physical flow. Again, this is a direct and natural consequence of the KMS condition.

VI. Discussion

From different aspects, the hereabove results appears to be compatible with some other approaches, in particular with the totally different context of superstrings theory. For example, we have already pointed out that in this context, J.J. Atick and E. Witten were the first to propose such an extension of the real temperature towards a complex domain [1]. Recently, in $N = 4$ supersymmetric string theory, I. Antoniadis, J.P. Deredinger and C. Kounnas [5] have also suggested to shift the real temperature to imaginary one by identification with the inverse radius of a compactified Euclidean time on S^1, with $R = 1/2\pi T$. Therefore, it appears quite natural to introduce a complex temperature in the thermal moduli space, the imaginary part coming from the $B_{\mu\nu}$ antisymmetric field under type $IIA \overset{S/T/U}{\longleftrightarrow}$ type $IIB \overset{S/T/U}{\longleftrightarrow}$ $Heterotic$ string-string dualities. More precisely, in Antoniadis *et al*., approach, the field controlling the temperature comes from the product of the real parts of three complex fields : $s = \mathrm{Re}S$, $t = \mathrm{Re}T$ and $u = \mathrm{Re}U$. Within our KMS approach, the imaginary parts of the moduli S, T, U can be interpreted in term of Euclidean temperature.

(ii) From another point of view, according to most of the models, supergravity is considered as broken for scales greater than the Planck scale. At this stage, we have emphasized the deep relation between supersymmetry breaking and the cancellation of the thermodynamic equilibrium state [13]. With this in mind, C. Kounnas has recently demonstrated that a five-dimensional ($N = 4$) supersymmetry can be described by a four-dimensional theory in which supersymmetry is spontaneously broken by finite thermal effects [6]. We have suggested in Ref. [13] to apply this scenario to our setting. Indeed, the end of the thermal equilibrium phase at the Planck scale might bring about the breaking of KMS state and of supersymmetry $N = 4$. This corresponds exactly, in our case, to the decoupling between imaginary time and real time.

To sum up, the chain of events able to explain the transition from the topological phase to the physical phase of the spacetime might be the following:

$$\{\text{thelmodynamical equilibrium breaking}\} \to \{\text{KMS state breaking}\}$$
$$\to \{\text{imaginary time / real time decoupling}\}$$
$$\to \{\text{topological state / physical state decoupling}\}$$
$$\to \{\text{Supersymmetry breaking}\}.$$

We have given a detailed description of such a transition in [13]. Likewise, the supersymmetry is broken in [5-6] by the finite temperature, which corresponds in our view to the *decoupling* between real (i.e. physical) and imaginary (i.e. topological) temperature. According to this supersymmetry breaking and transition from topological state to physical state might be deeply connected.

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