Octonions and Exceptional Groups?

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The basic properties of octonions and their relations to the exceptional groups are presented in a pedagogical manner. Among the topics discussed are a brief history and the basic properties of octonions, non-associative algebras, octonions and exceptional groups, and trends in using octonions in physics.

I. INTRODUCTION

There are growing trends in applying the exceptional groups to physics in the past decade. For example, certain grand unified theories are based on the exceptional group $E_6$. In addition, the heterotic string theory in 10 dimensions involves the gauge group $E_6 \times E_8$, but all the exceptional groups are intimately connected with octonions. If future developments in physics indeed involve exceptional groups, the octonions will certainly play some important roles in physics. The purpose of this talk is to present a pedagogical introduction to the basic properties of octonions and their relations to the exceptional groups. My understanding of the octonions and the exceptional groups comes, as a byproduct, from our studies of $E_6$ grand unified theory. Many of the results presented or reviewed here can be found from the original works of F. Gürsey, A. R. Dundarer, M. Koca and N. Ozdes, S. Okubo, R. Slansky, and J. L. Rosner.

The contents of the article are as follows. Sec. 2 contains a brief history of octonions. Basic properties of octonions are summarized in Sec. 3. Sec. 4 has an introductory description of non-associative algebra. The connection between octonions and exceptional groups is explained in Sec. 5. Recent trends in using octonions and exceptional groups in physics are emphasized in Sec. 6.

II. HISTORY OF OCTONIONS

The history of quaternions and octonions is described vividly by van der Waerden and van der Blij respectively. Only a brief description is presented here. The story begins with...
the discovery of quaternions by the famous mathematician and physicist W. R. Hamilton (1805–1865). He had made many important contributions in physics and mathematics. His work in mechanics is well-known among physicists today. I think all of us would agree with what Schrödinger has said: "The Hamiltonian principle has become the cornerstone of modern physics."

Hamilton’s other major discovery is the number system of quaternions in October 1843. Hamilton spent nearly 10 years vainly trying to construct a system of numbers with 3 units he called them triplets. In one of his letters to his son, he recalls how each morning, coming down to breakfast, his son would ask him: "Well, Papa, can you multiply triplets?" and he would sadly reply: "No, I can only add and subtract them." Finally, the solution of the problem which had occupied him for such a long time came to him in a flash. He gave up the idea of triplets and leaped into the fourth dimension. The quaternions and the fundamental formula for multiplication were discovered immediately.

A quaternion \( x \) is a set of 4 real numbers
\[
x = (a, b, c, d) = a + bi + cj + dk.
\]  
(2.1)

The norm of \( x \) is defined to be
\[
N(x) = a^2 + b^2 + c^2 + d^2.
\]  
(2.2)

The multiplication of quaternions is required to satisfy the norm condition
\[
N(xy) = N(x)N(y).
\]  
(2.3)

The fundamental formula for multiplication discovered by Hamilton are
\[
\begin{align*}
1^2 &= j^2 = k^2 = -1, \\
i j &= -ji = k, \\
j k &= -kj = i, \\
k i &= -ik = j.
\end{align*}
\]  
(2.4)

Note that the quaternion algebra is not commutative but it is still associative.

The octonion algebra was found by J. T. Graves in December 1843, two months after Hamilton’s discovery of quaternions. There are 8 unit elements for octonions: \( 1, i, j, k, l, m, n, o \). Graves defined their multiplication as
Hamilton noted to Graves on July 8, 1844, that the associative law does not hold for octonions. However, the paper by Graves was not published. The octonions were rediscovered by Cayley in 1845; they are also known as Cayley numbers. Cayley was a lawyer. He limited his law work to just enough to provide a livelihood and reserved the remainder of his time to do mathematics. During his lifetime he published 967 papers. In connection with Cayley, we should also mention that he created the theory of matrices.

It was proved by Hurwitz in 1898 that \( n = 1, 2, 4, 8 \) are the only possibilities for the norm condition:

\[
(a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2) = (c_1^2 + \cdots + c_n^2).
\]  

The formula for the sum of 4 squares

\[
(a_1^2 + \cdots + a_4^2)(b_1^2 + \cdots + b_4^2) = (c_1^2 + \cdots + c_4^2),
\]  

was discovered by Euler in 1748. The formula for the sum of 8 squares

\[
(a_1^2 + \cdots + a_8^2)(b_1^2 + \cdots + b_8^2) = (c_1^2 + \cdots + c_8^2),
\]  

was found by Degen in 1818. They were proved by Graves (1843) and Cayley (1845) by means of octonions. Legendre showed that it is impossible to define \((c_1, c_2, c_3)\) as a bilinear function of \((a_1, a_2, a_3)\) and \((b_1, b_2, b_3)\) such that the following identity holds:

\[
(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) = (c_1^2 + c_2^2 + c_3^2).
\]  

For example,

\[
3 = 1^2 + 1^2 + 1^2,
\]

\[
21 = 1^2 + 2^2 + 1^2.
\]
but $3 \times 21 = 63$ cannot be represented rationally as sum of $3$ squares. Fortunately Hamilton did not read Legendre: he was self-taught!

### III. BASIC PROPERTIES OF OCTONIONS

In this section, basic properties of octonions will be summarized. For a more complete description, please see Ref. 9. An octonion $x$ is a set of $8$ real numbers

$$x = (x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7)$$

$$= x_0e_0 + x_1e_1 + \ldots + x_7e_7,$$

where $e_0$ is the identity element and $e_i$ ($i = 1, 2, \ldots, 7$) are $7$ imaginary units. The basis vectors $e_0, e_1, \ldots, e_7$ satisfy the multiplication rule

$$e_ie_j = e_je_i = e_{ij} = \sum_k f_{ijk}e_k,$$

$$i, j, k = 1, 2, \ldots, 7$$

where $f_{ijk}$ are completely antisymmetric in $i, j, k$ with values $1, 0, -1$. A convenient basis has been chosen in physics literature such that

$$f_{123} = f_{246} = f_{357} = f_{572} = f_{714} = 1.$$  

A nice way of representing the multiplication table is provided by Fig. 1.

When $3$ imaginary units of octonions $e_a, e_b$ and $e_c$ satisfy the relation

$$e_a e_b e_c = -1 \quad (a \neq b \neq c),$$

we call them an associative triad. The basis chosen by Eq. (3.3) implies $7$ associative triads, e.g. $e_1 e_2 e_3 = -1$, etc.

The octonionic conjugate $\bar{x}$ of an octonion $x = x_0e_0 + x_1e_1 + \ldots + x_7e_7$, is defined as

$$\bar{x} = x_0e_0 - x_1e_1 - \ldots - x_7e_7,$$

where

$$\bar{e}_a = e_a, \bar{e}_i = -e_i.$$  

The norm $N(x)$ of an octonion $x$ is its scalar product with itself:

$$N(x) = \bar{x}x = x\bar{x} = x_0x_0.$$
The norm as defined above satisfies the norm condition:

\[ N(xy) = N(x)N(y), \]  

where \( x, y \) are 2 octonions.

An algebra \( A \) over a field \( F \) is a division algebra if \( A \) has a unit for multiplication and contains a multiplicative inverse of each nonzero element. It was proved by Hurwitz (1898) that over the field of real numbers there exists only 4 division algebras, i.e., the real numbers, complex numbers, quaternion algebras and octonion algebras. The octonion algebra is neither commutative nor associative. An algebra \( A \) is called non-associative if the associative law for multiplication does not hold for some elements of \( A \). Since \( (e_1e_2)e_3 \neq e_1(e_2e_3) \), one can easily see that the octonion algebra is non-associative. The associator \( (x, y, z) \) defined by

\[ (x, y, z) \equiv (xy)z - x(yz), \]  

measures the deviation of the algebra \( A \) from the associative law. If the associator is totally antisymmetric for exchanges of any 2 variables, i.e.,

\[ (x, y, z) = -(x, y, z) = -(y, x, z) = -(z, x, y), \]  

then the algebra is called \(^1\text{algebra}\). The octonion algebra is neither commutative nor associative. It belongs to the class of alternative algebras. Some further properties of octonions are:

(1) power associativity
\[ x^r x^s = x^{r+s}, \]
\[ (x^n x^m) x^r = x^n (x^m x^r), \]
so that the integral powers of an octonion are unambiguously defined. (2) Moufang identities: For any 3 octonions \( x, y \) and \( z \), we have
\[ (a) \quad (yx)(yz) = z(xy)z, \]
\[ (b) \quad (y)(x + yz) = y[x(yz)], \]
\[ (c) \quad z[y(xy)] = [(xy)z]y. \]

### IV. NON-ASSOCIATIVE ALGEBRA

An algebra \( A \) is called non-associative if the associative law for multiplication does not hold for some elements of \( A \). The most familiar example of a non-associative algebra widely used in physics is Lie algebra. Recall the following definition for a Lie algebra. Suppose that in addition to the conditions for an algebra, we have the following extra conditions:

1. Anti-commutativity
   \[ xy = -yx, \]  
2. Jacobi identity
   \[ (xy)z + (yz)x + (zx)y = 0, \] for any \( x, y, z \in A \), then \( A \) is called a Lie algebra. It is more customary to write \([x, y]\) instead of \( xy \) for such a case.

Another important example of a non-associative algebra is Jordan algebra. Set \( x^2 = xx \). If the product of the algebra satisfies the following extra conditions:

1. Commutativity
   \[ xy = yx, \]
2. Jordan identity
   \[ (x^2 y)z = x^2 (yz), \] then it is called a Jordan algebra. In this case, it is customary to write \( xe^y \) instead of \( xy \) for the product. Given an algebra \( A \) with the bilinear product \( xy \), we can define new products by

\[ [x, y] = xy - yx, \]
\[ x \cdot y = \frac{1}{2} (xy + yx), \]
The algebra with the product $[x, y]$ is called $A^-$ and the algebra with the product $x \cdot y$ is called $A^+$. If $A$ is an associative algebra, then $A^-$ is a Lie algebra and $A^+$ is a Jordan algebra since

$$[[x, y], z] + [[y, z], x] + [[x, z], y] = 0,$$

(4.7)

$$x^2 \cdot y \cdot x = x^2 \cdot (y \cdot x).$$

(4.8)

The classification of finite dimensional simple Jordan algebras was given by Jordan, von Neumann and Wigner in 1934. Their results can be summarized in the following theorem.

**Theorem:** If $A$ is a finite dimensional simple Jordan algebra, then $A$ must be obtained only in the following 2 ways:

1. The Jordan algebra can be obtained from some associative algebra. Such an algebra is known as a special Jordan algebra. There are 4 types of special Jordan algebras:
   - (I) $J(Q)$,
   - (II) $H_n(R)$,
   - (III) $H_n(C)$,
   - (IV) $H_n(Q)$,
   where $J(Q)$ comes from a quadratic form $Q$

$$J(Q) = R1 \oplus V,$$

(4.9)

$$z \cdot y = Q(z, y)$$

for $z, y \in V$,

where 1 is the unit element and $R$ denotes the real numbers. The meaning of $J(Q)$ can be understood more easily in terms of the following basis of a Jordan algebra

$$e_m = (1, e_1),$$

where

$$m = 0, 1, 2, \cdots, n,$$

$$i = 1, 2, \cdots, n,$$

$$e_o = 1$$

(4.10)

The basis can be chosen such that

$$e_i \cdot e_j = \delta_{ij} 1 + T_{ijk} e_k,$$

(4.11)

where $i, j, k = 1, 2, \cdots, n$.

Every element of a Jordan algebra can be written as a linear combination of $e_m$ with $m = 0, 1, \cdots, n$. The type I algebras $J(Q)$ are the simplest because Eq. (4.9) implies that $T_{ijk}$ vanishes for this particular type of algebra. They are the generalizations of the $\sigma$ matrices under anticom-
mutation and are therefore subalgebras of the Clifford algebras of positive definite quadratic forms. The remaining 3 types of special Jordan algebras $H_n(R)$, $H_n(C)$ and $H_n(Q)$ are realizable as $n \times n$ hermitian matrices over the real numbers $R$, complex numbers $C$ and quaternions $Q$ respectively, with the Jordan product again being the anticommutator.

(2) The Jordan algebra is a 27-dimensional algebra which can be constructed as follows:

Let

$$X = \begin{pmatrix} a_1 & x & \bar{y} \\ \bar{x} & a_2 & z \\ y & \bar{z} & a_3 \end{pmatrix},$$

(4.12)

where $a_1, a_2, a_3$ are real or complex numbers and $x, y, z$ are real or complex octonions, and $\bar{x}, \bar{y}, \bar{z}$ are the octonionic conjugates. Define the Jordan product by

$$X \cdot Y = \frac{1}{2}(XY + YX),$$

(4.13)

this then defines the 27-dimensional real or complex exceptional Jordan algebra. Note that $XY$ is a matrix product but it is not associative since its octonionic elements are not associative.

In the Jordan formulation of Quantum Mechanics the observables and density matrices representing a physical system are elements of a Jordan algebra. If the Jordan algebra is special, then the Jordan formulation is equivalent to the Dirac formulation of Quantum Mechanics in terms of commutators. In 1978, Gineygin, Piron, and Ruegg showed that it is possible to formulate Quantum Mechanics based on the exceptional Jordan algebra by the Jordan formulation. It is called octonionic Quantum Mechanics. The classification of infinite dimensional Jordan algebras has been accomplished by Zelmanov in 1983. It appears that all infinite dimensional simple Jordan algebras are extensions of special Jordan algebras. There are no infinite dimensional exceptional Jordan algebras. This implies that no Hilbert space formulation of octonionic Quantum Mechanics is possible. Thus octonionic Quantum Mechanics is a radical generalization of Quantum Mechanics and is very interesting!

V. OCTONIONS AND EXCEPTIONAL GROUPS

The exceptional Jordan algebra $A$ is intimately connected to the exceptional Lie groups: $G_2$ arises as the automorphism group of the octonions $0$, $F_4$ as the automorphism group of $A$, $E_6$ as the isotopies of $A$, while $E_7$ and $E_8$ can be built out of $A$ by more complicated means. Since the exceptional Jordan algebra can only be constructed from $3 \times 3$ matrices with entries in the octonions; there is an intimate connection between octonions and the exceptional groups. The main purpose of this talk is to explain this connection.
V-1. Automorphism Group of Octonions

An automorphism of an algebra \( A \) is defined as an isomorphism of \( A \) onto itself. Under an automorphism, the multiplication table of \( A \) is left invariant, i.e.

If \( x, y \in A, T \in \text{Aut} A \)

then

\[ T(xy) = T(x)T(y). \]  

(5.1)

The set of all automorphisms of composition algebras forms a group. The automorphism group of the quaternions is the SU(2) group. Below we shall explain that the automorphism group of the octonions is the exceptional Lie group \( G_2 \).

For each Cayley basis, there are 7 independent canonical automorphisms. Each canonical automorphism involves 2 independent parameters. Hence canonical automorphisms generate a 14-parameter Lie group. Thus the automorphism group of octonions is a 14-parameter Lie group of type \( G_2 \). A detailed derivation of the automorphism group of the octonions is given by G"unaydin and G"ursey.

V-2. Tits' Construction of Exceptional Groups

The automorphism group of a Jordan algebra can be easily obtained by noting that the Lie algebra of the automorphism group of the Jordan algebra is given by the derivation algebra of the Jordan algebra. Let us denote the Jordan algebra of \( n \times n \) hermitian matrices over the division algebra \( A \) by \( J_n^A \). The automorphism groups of \( J_n^R, J_n^C \) and \( J_n^Q \) are SO(\( n \)), SU(\( n \)) and Sp(2\( n \)), respectively. In this way, the classical groups are connected with real numbers, complex numbers and quaternions, respectively.

The automorphism group of the exceptional Jordan algebra \( J_3^O \) is \( F_4 \). This can be seen as follows. In this case, the observables \( X \) are elements of \( J_3^O \). They are 3 x 3 octonionic matrices of the form

\[ X = \begin{pmatrix} a_1 & z & \bar{y} \\ \bar{z} & a_2 & y \\ z & \bar{y} & a_3 \end{pmatrix}, \]  

(5.2)

where \( a_1, a_2, a_3 \) are real numbers and \( x, y, z \) are octonions with the bars denoting octonionic conjugation. Chevalley and Shafer showed that these observables can be transformed in terms of associators as

\[ X \rightarrow X' = E_{A,B} X = X + \frac{1}{11} (A, X, B) + \frac{1}{21} (A, (A, X, B), B) + \cdots. \]  

(5.3)
with A and B of the form Eq. (5.2) but with zero trace. Hence each of them has 27 - 1 = 26 parameters. Thus the group has 2 x 26 = 52 parameters and is the exceptional group $F_4$ acting on the exceptional observable $X$.

All the exceptional groups $G_2, F_4, E_6, E_7, E_8$ are connected with the octonions. Recall that $G_2$ is the automorphism group of the octonions and $F_4$ is the automorphism group of the exceptional Jordan algebra $J_3^0$. The exceptional groups $F_4, E_6, E_7,$ and $E_8$ can be constructed in a unified way out of the automorphism algebras of the Jordan algebras $J_3^i (i = 1, 2, 4, 8)$ and the automorphism algebras of the Hurwitz algebras $H_j (j = 1, 2, 4, 8)$ by means of the Tits construction. 24 According to the Tits construction, the exceptional groups $F_4, E_6, E_7,$ and $E_8$ fit in a square called the magic square of Table I with entries $J_3^i$ and $H_j (i, j = 1, 2, 4, 8)$. In Table I, the row below $J_3^i$ corresponds to the automorphism group of the Jordan algebras $J_3^i$. The column to the right of $H_j$ represents the automorphism groups of the Hurwitz algebras $H_j$. The group at the intersection $(ij)$ is associated with the Lie algebra $L_{ij}(i, j = 1, 2, 4, 8)$ given by

$$L_{ij} = \text{Aut} \ J_3^i + J_3^i H_j + \text{Aut} \ H_j,$$

where $J_3^i$ is the traceless 3 x 3 hermitian matrix $J_3^i$ and $H_j$ is the purely imaginary part of the Hurwitz number $H_j$. Thus, the dimension of the magic square group is

$$d(L_{ij}) = d(\text{Aut} \ J_3^i) + [d(J_3^i) - 1][d(H_j) - 1] + d(\text{Aut} \ H_j).$$  

Let's consider the following example.

**Ex:** The Lie algebra $L_4$ with $i = 2$ and $j = 8$.

Eq. (5.4) gives

$$L_{28} = \text{Aut} \ J_3^2 + J_3^2 H_8^8 + \text{Aut} \ H_8^8$$

$$= SU(3) + J_3^2 H_8^8 + G_2,$$

$$d(L_{28}) = d(SU(3)) + [d(J_3^2) - 1][d(H_8^8) - 1] + d(G_2)$$

$$= 8 + [9 - 1][8 - 1] + 14 = 78.$$

**TABLE I. Magic square.**

<table>
<thead>
<tr>
<th>$H_1$</th>
<th>$H_2$</th>
<th>$H_3$</th>
<th>$H_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_3^1$</td>
<td>$J_3^1$</td>
<td>$J_3^2$</td>
<td>$J_3^4$</td>
</tr>
<tr>
<td>$J_3^5$</td>
<td>$J_3^8$</td>
<td>$J_3^{10}$</td>
<td>$J_3^{12}$</td>
</tr>
<tr>
<td>$J_3^{14}$</td>
<td>$J_3^{16}$</td>
<td>$J_3^{18}$</td>
<td>$J_3^{20}$</td>
</tr>
<tr>
<td>$J_3^{22}$</td>
<td>$J_3^{24}$</td>
<td>$J_3^{26}$</td>
<td>$J_3^{28}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$J_3^1$</th>
<th>$J_3^2$</th>
<th>$J_3^4$</th>
<th>$J_3^8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SO(3)$</td>
<td>$SU(3)$</td>
<td>$Sp(6)$</td>
<td>$F_4$</td>
</tr>
<tr>
<td>$SU(3)$</td>
<td>$SU(3)$</td>
<td>$SU(6)$</td>
<td>$E_6$</td>
</tr>
<tr>
<td>$SO(12)$</td>
<td>$SO(12)$</td>
<td>$SO(12)$</td>
<td>$E_7$</td>
</tr>
<tr>
<td>$E_8$</td>
<td>$E_8$</td>
<td>$E_8$</td>
<td>$E_8$</td>
</tr>
</tbody>
</table>
which is the dimension of the Lie algebra of $E_6$.

This example shows that $E_6$ is the octonionic generalization of SU(3). Hence, Tits\textsuperscript{1} construction displays the exceptional groups $F_4, E_6, E_7$ and $E_8$ as the octonionic generalizations of $SO(3), SU(3), Sp(6)$ and $F_4$. For more details concerning the Tits\textsuperscript{1} construction and related matters, please see Ref. 8.

V-3. Integral Elements of Division Algebra

Any element $x$ of a division algebra satisfies a second order equation with real coefficients:

$$x^2 - (2Sc_x)x + N(x) = 0,$$

where $2Sc_x = x + x$ is twice the scalar part of $x$. The set of elements $A$ satisfying the above equation with integer coefficients, i.e. $x + x = integer$, and $x = integer$, are called integral elements of the 4 division algebras provided they obey the following conditions:

1. $A$ is closed under subtraction and multiplication;
2. $A$ contains $I$;
3. $A$ is not a subset of a larger set satisfying (1) and (2).

According to the Cayley-Dickson procedure,\textsuperscript{9,10} an octonion $x$ can be represented by a pair of quaternions $p$ and $q$:

$$x = [p, q] = p + e_7q.$$ (5.9)

This can be seen as follows:

$$x = (x_0e_0 + x_1e_1 + x_2e_2 + x_3e_3) + (x_4e_4 + x_5e_5 + x_6e_6 + x_7e_7)$$

$$= (x_0e_0 + x_1e_1 + x_2e_2 + x_3e_3) + e_7(x_4e_0 + x_5e_1 + x_6e_2 + x_7e_3)$$

$$= p + e_7q.$$ (5.10)

The following formulas are useful for this construction:

$$\bar{x} = [\bar{p}, \bar{q}] = [\bar{p}, -q],$$ (5.11)

$$N(x) = \bar{x}x = [p, q][p, q] = \bar{p}p + \bar{q}q,$$ (5.12)

$$[p, q][r, s] = [pr - sq, rq + ps].$$ (5.13)

As a matter of fact, we may define a complex number $C$ by pairing real numbers $a_0$ and $a_1$:

$$[a_0, a_1] = a_0 + e_1a_1 = C.$$ (5.14)

Similarly, we may define a quaternion $q$ by pairing complex numbers $C$ and $\bar{C}$.
\[ q = [C, C'] = C + \epsilon_3 C' \]
\[ = a_0 + \epsilon_1 a_1 + \epsilon_2 (a_3 + \epsilon_1 a_2) \]
\[ = a_0 + \epsilon_1 a_1 + \epsilon_2 a_2 + \epsilon_3 a_3. \]  

(5.15)

Then, by pairing quaternions \( p \) and \( q \), one may obtain an octonion as given by Eq. (5.9). Note that there does not exist any division algebra beyond octonions. Thus the Cayley-Dickson procedure cannot be applied to a pair of octonions.

The integral elements of division algebras have been nicely discussed by Koca and Ozdes. They showed that by starting with

\[ \pm 1 = \text{the integral elements of real numbers of unit norm,} \]
\[ = \text{the non-zero roots of SU(2)}, \]

and

\[ \pm 1/2 = \text{the weights of spinor representation of SU(2)}, \]

one can construct the unit norm integral elements of complex numbers, quaternions and octonions by the Cayley-Dickson procedure of pairing. It turns out that the unit norm integral elements of the complex numbers are \( \pm 1, \pm \epsilon_1 \). They are the non-zero roots of \( \text{SO}(4) \sim \text{SU}(2) \times \text{SU}(2) \). The unit norm integral elements of the quaternions are

\[ \pm 1, \pm \epsilon_1, \pm \epsilon_2, \pm \epsilon_3, \]
\[ \pm \frac{1}{2} \pm \frac{1}{2} \epsilon_1 \pm \frac{1}{2} \epsilon_2 \pm \frac{1}{2} \epsilon_3. \]

They are the non-zero roots of \( \text{SO}(8) \). There are 240 unit norm integral elements of the octonions. They were first obtained by Coxeter in 1946. He showed that a basis of 8 elements can be chosen such that the geometrical representation of this basis becomes Fig. 2. One can easily see it is the same as the Dynkin diagram of \( \text{E}_8 \). The 8 basis elements are, in fact, the simple roots of \( \text{E}_8 \). The 240 unit norm integral elements are simply the non-zero roots of \( \text{E}_8 \). It is amazing that the Dynkin diagram of \( \text{E}_8 \) first appeared in Coxeter's geometrical representation of 8 basis elements of integral octonions.

\[ \begin{array}{ccccccc}
\epsilon & -h & j & \epsilon h & 1 & \epsilon h & ke \\
\hline
& & & & & & \\
\hline
\end{array} \]

FIG. 2. Coxeter's geometrical representation of 8 basis elements of integral octonions.
sentation of integral octonions without any reference to group theory. Thus, instead of the Dynkin diagram, it is appropriate to call it, the Coxeter-Dynkin diagram. The 240 unit norm integral elements of octonions can be expressed in terms of 8 auxiliary octonions of quadratic norm $1/2$:

$$
\begin{align*}
    l_1 &= \frac{1}{2}(e_1 + e_4),
    l_2 &= \frac{1}{2}(e_2 + e_5),
    l_3 &= \frac{1}{2}(e_3 + e_6),
    l_6 &= \frac{1}{2}(1 + e_7),
    l_4 &= \frac{1}{2}(e_1 - e_4),
    l_5 &= \frac{1}{2}(e_2 - e_5),
    l_7 &= \frac{1}{2}(e_3 - e_6),
    17 &= \frac{1}{2}(1 - e_7).
\end{align*}
$$

(5.16)

The 240 integral elements consists of 112 elements of the form $\rho_r$, and 128 elements of the form $\sigma$:

$$
\begin{align*}
    \rho_r &= \pm l_r \pm s, r \neq s, \\
    \sigma &= \frac{1}{2}(\pm l_1 \pm l_2 \pm l_3 \pm l_4 \pm l_5 \pm l_6 \pm l_7 \pm l_8).
\end{align*}
$$

(5.17)

The unit norm integral elements of the division algebras are summarized in Table II.

V-4. Some Properties of $E_6$

The roots of $E_6$ can also be expressed in terms of the 8 auxiliary octonions of quadratic norm $1/2$ given by Eq. (5.16) plus a 9-th octonion $l_0$ of unit norm:

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<th>Numbers of associated integers</th>
<th>Associated group</th>
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<tr>
<td></td>
<td>$\sim SU(2) \times SU(2)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>quaterions</td>
<td>$\pm 1, \pm e_1, \pm e_2, \pm e_3$</td>
<td>24</td>
<td>SO(8)</td>
</tr>
<tr>
<td></td>
<td>$\pm 1/2 \pm 1/2 e_1 \pm 1/2 e_2 \pm 1/2 e_3$</td>
<td>240</td>
<td>$\mathbb{E}_8$</td>
</tr>
<tr>
<td>octonions</td>
<td>$\rho_r = \pm l_r \pm s (r \neq s)$</td>
<td>240</td>
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</tr>
<tr>
<td></td>
<td>$\sigma = 1/2(\pm l_1 \pm l_2 \pm l_3 \pm l_4 \pm l_5 \pm l_6 \pm l_7 \pm l_8)$</td>
<td>240</td>
<td></td>
</tr>
</tbody>
</table>
\[ l_0 = \frac{1}{2} (1 + e_1 + e_2 + e_3) \]  

(5.18)

Then the 72 non-zero roots of \( E_6 \) are given by\(^{25} \)

\[ \pm (l_\alpha - l_\beta), \quad \alpha, \beta = 0, 1, \cdots, 8. \]  

(5.19)

The 27 dimensional representation of \( E_6 \) can be expressed as a complex Jordan matrix

\[
J = \begin{pmatrix}
\alpha & c & b \\
\beta & a & \gamma \\
b & \bar{a} & \gamma
\end{pmatrix},
\]

(5.20)

where \( \alpha, \beta, \gamma \) are complex numbers and \( a, b, c \) are complex octonions. The infinitesimal transformation on \( J \) is given by

\[
J \rightarrow J' = T_{A,B,C}J = J + \frac{1}{2} (A, J, B) + iC \cdot J,
\]

(5.21)

where \( C \cdot J \) and \((A, J, B)\) denote the symmetric Jordan product and the associator respectively:

\[
C \cdot J = \frac{1}{2} (C J + J C),
\]

(5.22)

\[
(A, J, B) = (A \cdot J) \cdot B - A \cdot (J \cdot B) = -(B, J, A).
\]

(5.23)

In Eq. (5.21), \( A, B \) and \( C \) are traceless real Jordan matrices. Hence the total number of real parameters is given by \( r = 3 \times (27-1) = 78 \) parameters.

From the above discussions, one can see that the algebras and representations of \( E_6 \) can be worked out by using Jordan algebras and octonions. This method is very interesting. However, there exist various methods in dealing with \( E_6 \). Tensor methods based on the \( SU(3) \times SU(3) \times SU(3) \) subgroups of \( E_6 \) can be used in this case. It seems to me that for a group as complicated as \( E_6 \), the traditional tensor methods become quite cumbersome. In the following we shall give a simple discussion of \( E_6 \) by using Dynkin's method. This method is more general and easier to apply than the tensor method. For a more complete discussion, see the review article of Slansky.\(^{12} \)

The Dynkin diagram of \( E_6 \) is shown in Fig. 3. Since the rank of \( E_6 \) is equal to 6, there are 6 simple roots \( e_i (i = 1, 2, \cdots, 6) \). All of them are of equal lengths. The angle between 2 simple roots which are connected is 120°. The simple roots of \( E_6 \) can be expressed in terms of 7 Euclidean orthonormal vectors \( e_i (i = 1, 2, \cdots, 7) \):
\( \alpha_1 = e_1 - e_2, \)
\( \alpha_2 = e_2 - e_3, \)
\( \alpha_3 = e_3 - e_4, \)
\( \alpha_4 = e_4 - e_5, \)
\( \alpha_5 = e_5 - e_6, \)
\( \alpha_6 = \frac{1}{\sqrt{2}} (-e_1 - e_2 + e_3 + e_4 + e_5 + e_6) + \frac{1}{\sqrt{2}} e_7. \) (5.24)

One should compare this with Eq. (5.19). There the non-zero roots of \( E_6 \) are expressed in terms of octonions rather than Euclidean orthonormal vectors. \( E_6 \) is a rank 6 exceptional group with 78 parameters. There are 6 fundamental representations. Their Dynkin integers, dimensions and trialities are

- \((100000) - t = 1,\)
- \((000010)_{21^*} t = 2,\)
- \((000001) - t = 0,\)
- \((010000)_{251^*} t = 2,\)
- \((000100) - t = 1,\)
- \((001000) - t = 0.\)

For a more complete list of representations, see Table III. Note that \( E_6 \) is the only exceptional group which has complex representations. For example, the representations \( 27 \) and \( 27^* \) are inequivalent. If \( \lambda \) is a weight of \( 27 \), then \(-\lambda \) is not a weight of \( 27 \), but of \( 27^* \).

The grand unified theory based on \( E_6 \) is currently of great interest. It has many nice fea-
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tures: it is anomaly free; it has complex representations; the fermions of each generation belong to a single irreducible representation; it is the simplest generalization of the SU(5) and SO(10) models. Furthermore, there exists a superstring-inspired E₆ model. We have written several PC computer programs which will give us the quantum numbers of any representation of E₆. In this model, each generation of fermions is assigned to 27 of E₆. The quantum numbers of the first generation fermions are reproduced in Table IV. In addition to the known fermions which comprise the 5* plus 10 representations of SU(5), there exist in the first generation additional exotic fermions which make up the full representation 27 of E₆.

VI. TRENDS IN PHYSICS

We have seen that octonionic quantum mechanics based on exceptional Jordan algebra is very interesting. All exceptional groups G₂, F₄, E₆, E₇ and E₈ are intimately related to the octonions. The unit norm integral elements of octonions are associated with the root system of E₈. Hence the octonions may play an important role in physics. There is an often quoted saying of Dirac: "The God is the mathematician and a mathematically beautiful theory even with meager experimental supports will have a greater chance to be ultimately correct." The octonion algebra is certainly a beautiful mathematical entity. Furthermore, octonions are a natural

<table>
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<th>SO(10)</th>
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generalization of the real numbers, complex numbers, and quaternions. There exist only 4 types of number systems which satisfy the norm condition. Nature should make use of them. The present situation of octonions might remind us that Hamilton had devoted 20 years of his life to the theory of quaternions and their applications in physics without much success. One might conclude that it was a tragedy for Hamilton. He over-estimated their importance. It was not worthwhile for him to spend one-third of his precious life to the theory of quaternions. One would wonder then whether it is worthwhile to devote one's life to the study of octonions. After this talk I hope you are persuaded that the theory of quaternions and octonions will play some important roles in physics. They are relevant for future theoretical developments in physics. And, Hamilton is definitely ahead of his time.

ACKNOWLEDGMENTS

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REFERENCES