

Series-Expansions for Functions Satisfying Some Integral-Equations

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We describe a method for the construction of orthogonal and biorthogonal sets of functions which can be used for the expansion of functions satisfying

$$f(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \frac{\sin(u-t)a}{u-t} du$$

and related equations.

INTRODUCTION

WE describe a method for the construction of orthogonal and biorthogonal sets of functions which can be used for the expansion of functions satisfying

$$f(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \frac{\sin(u-t)a}{u-t} du$$

and related equations.

I. THE EQUATION

$$f(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \frac{\sin(u-t)a}{u-t} du$$

Because of the relation

$$\frac{1}{\pi} \frac{\sin(u-t)a}{u-t} = \frac{1}{2\pi} \int_{-a}^a e^{i(u-t)s} ds,$$

it is clear from the standpoint of the Fourier-integral, that a function satisfying

$$f(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \frac{\sin(u-t)a}{u-t} du$$

must be of the form

$$f(t) = \int_{-a}^a \varphi(\alpha) e^{i\alpha t} d\alpha.$$

This has been proved by Titchmarsh⁽¹⁾.

The series-expansions we obtain here for this equation, can therefore be applied to functions of this form only.

The method for the construction of orthogonal sets of functions consists in the following.

We expand $e^{i\alpha s}$ in a series of orthonormal functions $\varphi_n(s)$ satisfying

(1) E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (London, Oxford Press, 1950) P. 349.

$$\int_{-a}^a \varphi_n(s) \varphi_m(s) ds = \delta_{nm},$$

$$e^{ius} = \sum_n \varphi_n(s) \int_{-a}^a e^{iuv} \varphi_n(v) dv,$$

and find by substitution

$$\frac{1}{\pi} \frac{\sin(u-t)a}{u-t} = \frac{1}{2\pi} \sum_n \int_{-a}^a e^{-its} \varphi_n(s) ds \int_{-a}^a e^{iuv} \varphi_n(v) dv.$$

In this way we obtain a kind of bilinear formula for the kernel of the integral-equation.

Now it is easily seen that the functions $\int_{-a}^a e^{its} \varphi_n(s) ds$ form an orthogonal set. For when we compute

$$\begin{aligned} & \int_{-\infty}^{\infty} dt \int_{-a}^a e^{its} \varphi_n(s) ds \int_{-a}^a e^{-itv} \varphi_m(v) dv \\ &= \int_{-a}^a \varphi_n(s) ds \int_{-a}^a \varphi_m(v) dv \int_{-\infty}^{\infty} e^{it(s-v)} dt \\ &= 2\pi \int_{-a}^a \varphi_n(s) ds \int_{-a}^a \varphi_m(v) \delta(s-v) dv \\ &= 2\pi \int_{-a}^a \varphi_n(s) \varphi_m(s) ds \\ &= 2\pi \delta_{nm}, \end{aligned}$$

we see that the functions are orthogonal.

When we use a set satisfying

$$\int_{-a}^a r(s) \varphi_n(s) \varphi_m(s) ds = \delta_{nm},$$

we obtain a biorthogonal set.

We find

$$e^{ius} = \sum_n \varphi_n(s) \int_{-a}^a e^{iuv} \varphi_n(v) r(v) dv$$

and

$$\frac{1}{\pi} \frac{\sin(u-t)a}{u-t} = \frac{1}{2\pi} \sum_n \int_{-a}^a e^{-its} \varphi_n(s) ds \int_{-a}^a e^{iuv} \varphi_n(v) r(v) dv.$$

Here again the biorthogonal property is a direct consequence of the orthogonality of the $\varphi_n(s)$.

$$\begin{aligned} & \int_{-\infty}^{\infty} dt \int_{-a}^a e^{-its} \varphi_n(s) ds \int_{-a}^a e^{itv} \varphi_m(v) r(v) dv \\ &= 2\pi \int_{-a}^a \varphi_n(s) \varphi_m(s) r(s) ds \\ &= 2\pi \delta_{nm}. \end{aligned}$$

We find therefore a kind of eigenfunction-solutions of the integral-equation.

Here we make some applications of this theory.

At first we make the choice

$$\varphi_n(s) = e^{i \frac{n\pi}{a} s},$$

and find that

$$\begin{aligned} & \frac{1}{2} \int_{-a}^a e^{i(t - \frac{n\pi}{a})s} ds \\ &= \frac{\sin(t - \frac{n\pi}{a})a}{t - \frac{n\pi}{a}} \\ &= (-)^n \frac{\sin at}{t - \frac{n\pi}{a}} \end{aligned}$$

is a solution of

$$\begin{aligned} f(t) &= \frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \frac{\sin(u-t)a}{u-t} du \\ &= \frac{1}{a\pi} \int_{-\infty}^{\infty} f(u) \sum_n \frac{\sin at \sin au}{(t - \frac{n\pi}{a})(u - \frac{n\pi}{a})} du. \end{aligned}$$

The normalization is found to be

$$\begin{aligned} & \int_{-\infty}^{\infty} \frac{\sin^2 at}{(t - \frac{n\pi}{a})(t - \frac{m\pi}{a})} dt \\ &= (-)^{n+m} \frac{\pi}{2} \int_{-a}^a e^{i \frac{\pi}{a} (m-n)t} dt \\ &= a\pi \delta_{nm}. \end{aligned}$$

Then for any function which can be expanded into a series we get

$$f(t) = \frac{1}{a\pi} \sum_n \frac{\sin at}{t - \frac{n\pi}{a}} \int_{-\infty}^{\infty} f(u) \frac{\sin au}{u - \frac{n\pi}{a}} du.$$

$f(t)$ must satisfy

$$f(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \frac{\sin(u-t)a}{u-t} du,$$

therefore

$$\begin{aligned} & \frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \frac{\sin au}{u - \frac{n\pi}{a}} du \\ &= \frac{(-)^n}{\pi} \int_{-\infty}^{\infty} f(u) \frac{\sin(u - \frac{n\pi}{a})a}{u - \frac{n\pi}{a}} du \\ &= (-)^n f\left(\frac{n\pi}{a}\right) \end{aligned}$$

and so we get

$$f(t) = \sum_n f\left(\frac{n\pi}{a}\right) \frac{\sin(at - n\pi)}{at - n\pi} \quad (\text{Shannon's Theorem}).$$

As a second example we take

$$\varphi_n(s) = \sqrt{n + \frac{1}{2}} P_n(s)$$

and $a=1$.

Here we can make use of the well-known integral-representation

$$\begin{aligned} & \int_{-1}^1 e^{-its} P_n(s) ds \\ &= (-i)^n \left(\frac{2\pi}{t}\right)^{1/2} J_{n+1,2}(t), \end{aligned}$$

and obtain the well-known result

$$\frac{\sin(u-t)}{u-t} = \frac{\pi}{\sqrt{ut}} \sum_n \left(n + \frac{1}{2}\right) J_{n+1,2}(u) J_{n+1,2}(t).$$

The orthonormality-relation for these functions takes the form

$$\int_{-\infty}^{\infty} J_{n+1,2}(t) J_{m+1,2}(t) \frac{dt}{t} = \frac{\delta_{nm}}{n + \frac{1}{2}}$$

The expansion-theorem takes the form

$$f(t) = \sum_n \left(n + \frac{1}{2}\right) t^{-1} J_{n+1,2}(t) \int_{-\infty}^{\infty} f(u) J_{n+1,2}(u) \frac{du}{\sqrt{u}}.$$

To give an example of a biorthogonal system, we take the non-normalized Gegenbauer-polynomials $C_n^\nu(s)$ satisfying

$$\begin{aligned} & \int_{-1}^1 (1-s^2)^{\nu-1} {}_2C_n^\nu(s) C_n^\nu(s) ds \\ &= \delta_{nm} \frac{\pi \Gamma(n+2\nu)}{2^{2\nu-1} (n+\nu) n! (\Gamma(\nu))^2}. \end{aligned}$$

Here we can use the integral-representation

$$J_{n+\nu}(u) = \frac{(-i)^n \Gamma(2\nu) n! \left(\frac{u}{2}\right)^\nu}{\Gamma(\nu + \frac{1}{2}) \Gamma(\frac{1}{2}) \Gamma(n+2\nu)} \int_{-1}^1 e^{ius} (1-s^2)^{\nu-1} {}_2C_n^\nu(s) ds,$$

and find

$$\frac{1}{\pi} \frac{\sin(u-t)}{u-t} = \frac{2^{\nu-1} \Gamma(\nu)}{\pi} \sum_n (n+\nu) i^n \frac{J_{n+\nu}(u)}{u^\nu} \int_{-1}^1 C_n^\nu(s) e^{-its} ds.$$

Here $\frac{J_{n+\nu}(t)}{t^\nu}$ and $\int_{-1}^1 C_n^\nu(s) e^{-its} ds$ form a biorthogonal system.

The expansion-theorem takes the form

$$f(t) = \frac{2^\nu \Gamma(\nu)}{2\pi} \sum_{n=0}^{\infty} (n+\nu) i^n \int_{-\infty}^{\infty} f(u) J_{n+\nu}(u) \frac{du}{u^\nu} \int_{-1}^1 C_n^\nu(s) e^{-its} ds$$

or

$$f(t) = \frac{2^\nu \Gamma(\nu)}{2\pi} \sum_{n=0}^{\infty} (n+\nu) i^n \frac{J_{n+\nu}(t)}{t^\nu} \int_{-\infty}^{\infty} f(u) du \int_{-1}^1 C_n^\nu(s) e^{-ius} ds.$$

By making use of

$$f(t) = \int_{-1}^1 \varphi(\alpha) e^{i\alpha t} d\alpha$$

and the integral-representations

$$C_{2n}^\nu(s) = \frac{(-)^n}{2^{\nu-1}\Gamma(\nu)} \int_0^\infty \cos(ts) J_{2n+\nu}(t) t^{\nu-1} dt; \quad -1 < \nu < \frac{3}{2}, \quad n \geq 1$$

$$C_{2n+1}^\nu(s) = \frac{(-)^n}{2^{\nu-1}\Gamma(\nu)} \int_0^\infty \sin(ts) J_{2n+\nu+1}(t) t^{\nu-1} dt; \quad -1 < \nu < \frac{3}{2}, \quad n \geq 0$$

one can rewrite the last equation in the form

$$f(t) = 2^\nu \Gamma(\nu+1) \frac{J_\nu(t)}{t^\nu} f(0) + \sum_{\nu=1}^\infty (2n+\nu) \frac{J_{2n+\nu}(t)}{t^\nu} \int_0^\infty (f(u) + f(-u)) J_{2n+\nu}(u) u^{\nu-1} du$$

$$+ \sum_{n=0}^\infty (2n+\nu+1) \frac{J_{2n+\nu+1}(t)}{t^{\nu+1}} \int_0^\infty (f(u) - f(-u)) J_{2n+\nu+1}(u) u^{\nu-1} du.$$

By making $\nu = \frac{1}{2}$, we come back to the case of the Legendre-polynomials.

Another interesting special case arises for the Tschebichef-polynomials $T_n(s)$ defined by

$$T_n(\cos \vartheta) = \cos n\vartheta$$

and satisfying the orthonormality-relation

$$\int_{-1}^1 T_n(s) T_m(s) \frac{ds}{\sqrt{1-s^2}} = \frac{\pi}{(2)} \delta_{nm}.$$

Here and in the following the symbol (2) stands for the number 2 when $n > 0$ and for the number 1 when $n = 0$.

We can make use of the integral-representation

$$J_n(u) = \frac{(-i)^n}{\pi} \int_{-1}^1 e^{i u s} T_n(s) \frac{ds}{\sqrt{1-s^2}},$$

and find

$$\frac{1}{A} \frac{\sin(u-t)}{u-t} = \frac{1}{2\pi} \sum (2) i^n J_n(u) \int_{-1}^1 e^{-i t s} T_n(s) ds$$

$$= \frac{1}{2\pi} \sum (2) i^n J_n(u) K_n(t)$$

$$= \frac{1}{2\pi} \sum (2) i^n J_n(t) K_n(u).$$

Here the biorthogonal system is formed by $J_n(t)$ and $K_n(t)$.

$$K_n(t) = \int_{-1}^1 e^{-i t s} T_n(s) ds.$$

The normalization-integral can be computed in the usual way.

$$\begin{aligned} & \int_{-\infty}^{\infty} J_n(t) K_m(t) dt \\ & \int_{-\infty}^{\infty} J_n(t) dt \int_{-1}^1 e^{-its} T_m(s) ds \\ & = 2\pi \frac{i^{-n}}{\pi} \int_{-1}^1 T_n(s) ds \int_{-1}^1 \frac{T_m(u) \delta(u-s)}{\sqrt{1-u^2}} du \\ & = i^{-n} \pi \delta_{nm} \begin{cases} 1; & n \neq 0 \\ 2; & n = 0. \end{cases} \end{aligned}$$

The expansion-theorem takes the form

$$f(t) = \frac{1}{2\pi} \sum_{n=0}^{\infty} (2) i^n K_n(t) \int_{-\infty}^{\infty} f(u) J_n(u) du$$

or

$$f(t) = \frac{1}{2\pi} \sum_{n=0}^{\infty} (2) i^n J_n(t) \int_{-\infty}^{\infty} f(u) K_n(u) du.$$

Here we consider the last expression a little further. We remember that $f(t)$ must be of the form

$$f(t) = \int_{-1}^1 e^{iat} \varphi(a) da,$$

and compute

$$\begin{aligned} & \int_{-\infty}^{\infty} f(u) K_n(u) du = \int_{-\infty}^{\infty} K_n(u) du \int_{-1}^1 \varphi(a) e^{iua} da \\ & = \int_{-1}^1 \varphi(a) da \int_{-1}^1 T_n(s) ds \int_{-\infty}^{\infty} e^{iu(a-s)} du \\ & = 2\pi \int_{-1}^1 \varphi(a) da \int_{-1}^1 T_n(s) \delta(a-s) ds \\ & = 2\pi \int_{-1}^1 \varphi(s) T_n(s) ds. \end{aligned}$$

The expansion takes the form

$$\begin{aligned} f(t) & = \int_{-1}^1 \varphi(a) e^{iat} da \\ & = \sum (2) i^n J_n(t) \int_{-1}^1 \varphi(s) T_n(s) ds. \end{aligned}$$

For an odd function $f(t)$, the function $\varphi(a)$ is also odd and we get

$$\begin{aligned} f(t) & = \int_{-1}^1 \varphi(a) e^{iat} da \\ & = 2i \sum_{n=0}^{\infty} (-)^n J_{2n+1}(t) \int_{-1}^1 \varphi(s) T_{2n+1}(s) ds, \\ & \quad (\varphi(-s) = -\varphi(s)). \end{aligned}$$

$J_{2n+1}(t)$

* E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (London, Oxford Press, 1950) p. 352.

$$f(t) = \sum_{n=0}^{\infty} (4n+2) J_{2n+1}(t) \int_0^{\infty} f(u) J_{2n+1}(u) \frac{du}{u},$$

when it satisfies the integral equation

$$f'(t) = \frac{1}{2} \int_0^{\infty} [f(u+t) + f(u-t)] \frac{J_1(u)}{u} du,$$

and it is easily seen that

$$f(t) = \int_{-1}^1 \varphi(\alpha) e^{i\alpha t} dt$$

with odd $\varphi(\alpha)$ satisfies this equation. For one finds with

$$\begin{aligned} f(t) &= \int_{-1}^1 \varphi(\alpha) e^{i\alpha t} d\alpha, \quad \varphi(-\alpha) = -\varphi(\alpha) \\ f'(t) &= i \int_{-1}^1 \alpha \varphi(\alpha) e^{i\alpha t} d\alpha \\ &= \frac{1}{2} \int_0^{\infty} [f(u+t) + f(u-t)] \frac{J_1(u)}{u} du \\ &= \frac{1}{2} \int_0^{\infty} \frac{J_1(u)}{u} du \int_{-1}^1 \varphi(\alpha) [e^{i(u+t)\alpha} + e^{i(u-t)\alpha}] d\alpha \\ &= 2i \int_0^1 \varphi(\alpha) \cos(i\alpha) d\alpha \int_0^{\infty} \sin(u\alpha) J_1(u) \frac{du}{u} \\ &= 2i \int_0^1 \alpha \varphi(\alpha) \cos(t\alpha) d\alpha \\ &= i \int_0^1 \alpha \varphi(\alpha) e^{i t \alpha} d\alpha = f'(t). \end{aligned}$$

Therefore the two expansions

$$\begin{aligned} f(t) &= \int_{-1}^1 \varphi(\alpha) e^{i t \alpha} d\alpha \\ &= 2i \sum_{n=0}^{\infty} (-)^n J_{2n+1}(t) \int_{-1}^1 \varphi(s) T_{2n+1}(s) ds \end{aligned}$$

and

$$f(t) = \int_{-1}^1 \varphi(\alpha) e^{i t \alpha} d\alpha = \sum_{n=0}^{\infty} (4n+2) J_{2n+1}(t) \int_0^{\infty} f(u) J_{2n+1}(u) \frac{du}{u}$$

should correspond to the same expansion. That is really the case. For one has with odd $\varphi(\alpha)$

$$\begin{aligned} \int_0^{\infty} f(u) J_{2n+1}(u) \frac{du}{u} &= \int_0^{\infty} J_{2n+1}(u) \int_{-1}^1 \varphi(\alpha) e^{i u \alpha} d\alpha \\ &= i \int_{-1}^1 \varphi(\alpha) d\alpha \int_0^{\infty} \sin(u\alpha) J_{2n+1}(u) \frac{du}{u} \\ &= i \frac{(-)^n}{2n+1} \int_{-1}^1 \varphi(s) T_{2n+1}(s) ds \end{aligned}$$

SO that

$$\begin{aligned} f(t) &= \int_{-1}^1 \varphi(a) e^{iat} dt \\ &= \sum_{n=0}^{\infty} (4n+2) J_{2n+1}(t) \int_0^{\infty} f(u) J_{2n+1}(u) \frac{du}{u} \\ &= 2i \sum_{n=0}^{\infty} (-)^n J_{2n+1}(t) \int_{-1}^1 \varphi(s) T_{2n+1}(s) ds; \quad \varphi(-a) = -\varphi(a). \end{aligned}$$

Here we mention some properties of the functions $K_n(t)$. They satisfy the recurrence-relations

$$\begin{aligned} K_{n+1}(t) + K_{n-1}(t) &= 2iK'_n(t) \\ 2 \operatorname{cost} &= K_0(t) - itK_1(t) \\ \operatorname{sint} &= 2iK_1(t) + \frac{t}{2}K_2(t) \\ 2 \operatorname{cost} &= (2n+1)(2n+3)K_{2n+2}(t) \\ &\quad - \frac{it}{2} [(2n+3)K_{2n+1}(t) - (2n+1)K_{2n+3}(t)]; \quad n=0, 1, 2, \dots \\ -2 \operatorname{sint} &= i 2n(2n+2)K_{2n+1}(t) \\ &\quad - \frac{t}{2} [(2n+2)K_{2n}(t) - 2nK_{2n+2}(t)]; \quad n=1, 2, 3, \dots \end{aligned}$$

Further one has for them the differential equation

$$t^2 K''_n(t) + 3K'_n(t) + (t^2 - (n^2 - 1))K_n(t) = \begin{cases} 2 \operatorname{cost}; & \text{even } n \\ -2i \operatorname{sint}; & \text{odd } n. \end{cases}$$

From this equation one obtains explicit expressions of the $K_n(t)$

$$\begin{aligned} K_{2n}(t) &= 2 \frac{\operatorname{sint}}{t} \left[1 + \sum_{\nu=1}^n (-)^\nu \frac{2^{2\nu} (2\nu)!}{(4\nu)!} \frac{2n(2n+2\nu-1)!}{(2n-2\nu)!} t^{-2\nu} \right] \\ &\quad + 2 \frac{\operatorname{cost}}{t} \sum_{\nu=0}^{n-1} (-)^\nu \frac{2^{2\nu+1} (2\nu+1)!}{(4\nu+2)!} \frac{2n(2n+2\nu)!}{(2n-2\nu+1)!} t^{-(2\nu+1)}, \\ K_{2n+1}(t) &= 2i \frac{\operatorname{cost}}{t} \left[1 + \sum_{\nu=1}^n (-)^\nu \frac{2^{2\nu} (2\nu)!}{(4\nu)!} \frac{(2n+1)(2n+2\nu)!}{(2n-2\nu+1)!} t^{-2\nu} \right] \\ &\quad + 2i \frac{\operatorname{sint}}{t} \sum_{\nu=0}^{n-1} (-)^\nu \frac{2^{2\nu+1} (2\nu+1)!}{(4\nu+2)!} \frac{(2n+1)(2n+2\nu+1)!}{(2n-2\nu)!} t^{-(2\nu+1)}. \end{aligned}$$

In our next example, we consider an even function $f(t)$.

In this case our equation takes the form

$$\begin{aligned} f(t) &= \frac{1}{\pi} \int_0^{\infty} f(u) \left[\frac{\sin(u-t)a}{u-t} + \frac{\sin(u+t)a}{u+t} \right] du \\ &= \frac{2}{\pi} \int_0^{\infty} f(u) du \int_0^a \cos(ts) \cos(us) ds. \end{aligned}$$

When we here expand $\cos(ts)$ in a series of orthogonal functions satisfying

$$\int_0^a \varphi_n(s) \varphi_m(s) ds = \delta_{nm},$$

we obtain

$$\begin{aligned} & \frac{1}{\pi} \left[\frac{\sin(u-t)a}{u-t} + \frac{\sin(u+t)a}{u+t} \right] \\ &= \frac{2}{\pi} \sum \int_0^a \varphi_n(s) \cos(us) ds \int_0^a \varphi_n(v) \cos tv dv \end{aligned}$$

and can easily show the orthogonality of these functions,

$$\begin{aligned} & \int_0^\infty dt \int_0^a \varphi_n(s) \cos(ts) ds \int_0^a \varphi_m(v) \cos tv dv \\ &= \int_0^a \varphi_n(s) ds \int_0^a \varphi_m(v) dv \int_0^\infty \cos(ts) \cos(tv) dt \\ &= \frac{\pi}{2} \int_0^a \varphi_n(s) ds \int_0^a \varphi_m(v) \delta(s-v) dv \\ &= \frac{\pi}{2} \delta_{nm}, \end{aligned}$$

and the equation takes the form

$$f(t) = \frac{2}{\pi} \int_0^\infty f(u) du \sum \int_0^a \varphi_n(s) \cos(us) ds \int_0^a \varphi_n(v) \cos(tv) dv$$

so that

$$\int_0^a \varphi_n(s) \cos(ts) ds$$

is a solution of this equation.

We consider as an example the functions $\varphi_n(s) = \cos \lambda_n s$ where the λ_n 's are the positive roots of

$$\lambda \operatorname{tg} \lambda a = a$$

with a constant a

One finds then

$$\begin{aligned} & \frac{1}{\pi} \left[\frac{\sin(u-t)a}{u-t} + \frac{\sin(u+t)a}{u+t} \right] \\ &= \frac{2}{\pi} \sum \frac{2 \cos \lambda_n a \cos ta (t \operatorname{tg} ta - a)}{a \left(1 + \frac{\sin 2\lambda_n a}{2\lambda_n a}\right) (t^2 - \lambda_n^2)} \int_0^a \cos us \cos \lambda_n s ds \end{aligned}$$

and

$$f(t) = \frac{2}{\pi} \int_0^\infty \sum \frac{2 \cos \lambda_n a \cos ta (t \operatorname{tg} ta - a)}{a \left(1 + \frac{\sin 2\lambda_n a}{2\lambda_n a}\right) (t^2 - \lambda_n^2)} f(u) du \int_0^a \cos us \cos \lambda_n s ds.$$

The integral over u is clearly $f(\lambda_n)$ and the result can be written in the form

$$f(t) = \frac{2}{a} \sum_n \frac{f(\lambda_n) \cos \lambda_n a}{1 + \frac{\sin 2\lambda_n a}{2\lambda_n a}} \frac{t \operatorname{tg} ta - a}{t^2 - \lambda_n^2} \cos ta.$$

For an odd function $f(t)$, we find in a quite similar way,

$$f(t) = \frac{1}{\pi} \int_0^\infty f(u) \left[\frac{\sin(u-t)a}{u-t} - \frac{\sin(u+t)a}{u+t} \right] du$$

$$= \frac{2}{\pi} \int_0^\infty f(u) du \int_0^a \sin ts \sin us ds.$$

When we here expand in a series of functions $\sin \mu_n t$, where the μ_n 's are the positive roots of

$$\mu \operatorname{ctg} \mu a = \beta$$

with a constant β , we find

$$f(t) = \frac{2}{a} \sum_n \frac{f(\mu_n) \sin \mu_n a}{1 - \frac{\sin 2\mu_n a}{2\mu_n a}} \frac{\beta - t \operatorname{ctg} ta}{t^2 - \mu_n^2} \sin ta$$

Here the functions

$$\frac{\beta - t \operatorname{ctg} ta}{t^2 - \mu_n^2} \sin ta$$

form an orthogonal set.

II. A REMARK REGARDING SHANNON' S THEOREM

Because Shannon' s theorem represents the function $f(t)$ for $t = \frac{n\pi}{a}$, even when $f(t)$ does not satisfy

$$f(t) = \frac{1}{\pi} \int_{-\infty}^\infty f(u) \frac{\sin(u-t)a}{u-t} du$$

we may expect that the function is approximated in some meaning by Shannon' s formula. To see what happens in such a case. we take as an example the function

$$f(t) = e^{-\alpha|t|}$$

and obtain by substitution in Shannon' s formula

$$s(t) = \frac{1}{2a} \sum_{n=-\infty}^\infty e^{-\alpha \frac{|n|}{\pi}} \int_{-a}^a e^{i(\frac{n\pi}{a}-t)s} ds$$

$$= \frac{1}{2a} \int_{-a}^a e^{-i t s} \sum_{n=0}^\infty (2) e^{-\alpha \frac{n\pi}{a}} \cos \frac{n\pi}{a} s ds.$$

Here the sum can easily be computed and we obtain

$$s(t) = \frac{1}{2\pi} \int_{-\pi}^\pi \frac{e^{i \frac{a}{\pi} t s} (1 - e^{-\frac{2\pi}{a}})}{1 + e^{-\alpha \frac{2\pi}{a}} - 2e^{-\alpha \frac{\pi}{a}} \cos s} ds$$

Here we introduce the variable $z = e^{is}$ and find

$$s(t) = \frac{1}{2\pi i} \int \frac{(e^{\frac{\pi\alpha}{a}} - e^{-\frac{\pi\alpha}{a}}) z^{\frac{a}{\pi} t}}{(e^{\frac{\pi\alpha}{a}} - z)(z - e^{-\frac{\pi\alpha}{a}})} dz$$

The contour is the unit-circle in the complex z -plane, is however not closed in general. When we want to compute the integral with Cauchy' s theorem, we

have to add to the contour a loop enclosing the negative real axis in the complex z-plane.

Then we find

$$s(t) = e^{-\alpha|t|} + \frac{\sin \alpha|t|}{\pi} \int_0^1 \frac{(e^{\frac{\pi\alpha}{a}} - e^{-\frac{\pi\alpha}{a}}) r^{\frac{a}{\pi} t}}{1 + 2(e^{\frac{\pi\alpha}{a}} + e^{-\frac{\pi\alpha}{a}}) r + r^2} dr.$$

The additional term vanishes identically only for $\alpha=0$ and in this case the function satisfies the integral equation. In general the expansion oscillates about the smooth curve $f(t) = e^{-\alpha|t|}$ and cuts it only for the values $t = \frac{n\pi}{a}$. In the limit $a \rightarrow \infty$, the additional term approaches the value zero and then we have to do with the Fourier-integral.

III. THE EQUATION

$$f(t) = \int_0^\infty u f(u) du \int_0^1 s J_\nu(ts) J_\nu(us) ds$$

In the case of the Hankel integral theorem, we find quite similar relations. The equation corresponding to that considered above is here

$$\begin{aligned} f(t) &= \int_0^\infty u f(u) du \int_0^1 s J_\nu(ts) J_\nu(us) ds \\ &= \int_0^\infty u f(u) \frac{u J_{\nu+1}(u) J_\nu(t) - t J_{\nu+1}(t) J_\nu(u)}{u^2 - t^2} du \end{aligned}$$

Here it is clear that $f(t)$ should be of the form

$$f(t) = \int_0^1 \varphi(a) J_\nu(ta) da$$

and a simple computation confirms this statement.

For the computation of the kernel, we can use the well-known formula.

$$\frac{J_{\nu+1}(z)}{J_\nu(z)} = -2z \sum_n \frac{1}{z^2 - j_n^2}$$

where the j_n 's are the positive roots of

$$J_\nu(z)$$

$$\begin{aligned} &\int_0^1 s J_\nu(su) J_\nu(st) ds \\ &= 2 \sum_n \frac{j_n^2 J_\nu(t) J_\nu(u)}{(t^2 - j_n^2)(u^2 - j_n^2)} \end{aligned}$$

$$\left(\frac{J_\nu(t)}{j_n^2 - t^2} \right)'_s$$

$$\frac{J_\nu(t)}{j_n^2 - t^2} = \frac{1}{j_n J_{\nu+1}(j_n)} \int_0^1 s J_\nu(ts) J_\nu(j_n s) ds.$$

Now we compute

$$\begin{aligned} & \int_0^\infty t dt \int_0^1 s J_\nu(ts) J_\nu(j_n s) ds \int_0^1 u J_\nu(tu) J_\nu(j_m u) du \\ &= \int_0^1 s J_\nu(j_n s) ds \int_0^1 J_\nu(j_m u) \delta(u-s) du \\ &= \int_0^1 s J_\nu(j_n s) J_\nu(j_m s) ds \\ &= \frac{J_{\nu+1}(j_n)}{2} \delta_{nm}. \end{aligned}$$

Thus we find the orthonormality-relation

$$\int_0^\infty \frac{t J_\nu^2(t) dt}{(j_n^2 - t^2)(j_m^2 - t^2)} = \frac{\delta_{nm}}{2j_n^2}.$$

The integral-equation takes therefore the form

$$f(t) = 2 \sum \frac{j_n^2 J_\nu(t)}{j_n^2 - t^2} \int_0^\infty \frac{u f(u) J_\nu(u)}{j_n^2 - u^2} du.$$

Because of

$$f(t) = \int_0^\infty u f(u) du \int_0^1 s J_\nu(us) J_\nu(ts) ds,$$

the last equation can be written in the form

$$\begin{aligned} f(t) &= 2 \sum \frac{j_n J_\nu(t)}{(j_n^2 - t^2) J_{\nu+1}(j_n)} \int_0^\infty u f(u) du \int_0^1 s J_\nu(us) J_\nu(j_n s) ds \\ &= 2 \sum \frac{j_n f(j_n) J_\nu(t)}{J_{\nu+1}(j_n) (j_n^2 - t^2)}, \end{aligned}$$

and that is essentially Takizawa's* result.

Instead of using the functions $J_\nu(j_n s)$ where the j_n 's are the positive zeros of $J_\nu(z)$, we can, if we like, also use the functions $J_\nu(\lambda_n s)$ where the λ_n 's are the positive roots of

$$\lambda J'_\nu(\lambda a) + h J_\nu(\lambda a) = 0.$$

The resulting formula takes the form

$$\begin{aligned} f(t) &= \sum \frac{2\lambda_n^2 f(\lambda_n)}{a^2[\lambda_n^2 + h^2 - \frac{\nu^2}{a^2}] J_\nu^2(\lambda_n a)} \int_0^a s J_\nu(ts) J_\nu(\lambda_n s) ds \\ &= \sum \frac{2\lambda_n^2 f(\lambda_n)}{a^2[\lambda_n^2 + h^2 - \frac{\nu^2}{a^2}] J_\nu^2(\lambda_n a)} \frac{t J'_\nu(at) + h J_\nu(at)}{\lambda_n^2 - t^2}. \end{aligned}$$

Here the functions

$$\frac{t J'_\nu(at) + h J_\nu(at)}{\lambda_n^2 - t^2}$$

form an orthogonal set which can be shown quite generally in the following way

* E. I. Takizawa, Keiko Kobayasi and Jenn-Lin Hwang, Chinese J. Phys. 5, 21, (1967).

In the kernel

$$\int_0^a s J_\nu(us) J_\nu(ts) ds$$

we expand $J_\nu(us)$ in a series of orthogonal functions $\varphi_n(s)$ satisfying

$$\int_0^a r(s) \varphi_n(s) \varphi_m(s) ds = \delta_{nm},$$

then we get

$$\begin{aligned} & \int_0^a s J_\nu(us) J_\nu(ts) ds \\ &= \sum_n \int_0^a s J_\nu(ts) \varphi_n(s) ds \int_0^a J_\nu(uv) \varphi_n(v) r(v) dv. \end{aligned}$$

Now

$$\begin{aligned} & \int_0^\infty t dt \int_0^a s J_\nu(ts) \varphi_n(s) ds \int_0^a r(v) J_\nu(tv) \varphi_m(v) dv \\ &= \int_0^a r(v) \varphi_m(v) dv \int_0^a \varphi_n(s) \delta(s-v) ds \\ &= \int_0^a r(s) \varphi_n(s) \varphi_m(s) ds \\ &= \delta_{nm}. \end{aligned}$$

For $r(s)=s$, we obtain an orthogonal set.

Our theory leads to simple relations, when we use for the expansion of the Bessel functions the Jakobi-polynomials,

$$F_n(\nu+1, \nu+1, s) = F(-n, n+\nu+1, \nu+1, s).$$

They satisfy the differential equation

$$\frac{d}{ds} [s^{\nu+1}(1-s) \frac{dF_n}{ds}] + n(n+\nu+1)s^\nu F_n = 0$$

and the orthonormality-relation

$$\begin{aligned} & \int_0^1 s^\nu F_n(\nu+1, \nu+1, s) F_m(\nu+1, \nu+1, s) ds \\ &= \left[\frac{n! \Gamma(\nu+1)}{\Gamma(n+\nu+1)} - \frac{2}{2n+\nu+1} \delta_{nm} \right]. \end{aligned}$$

We use these functions in the following way. In the interval $0 \leq s \leq 1$, we expand $s^{-\frac{\nu}{2}} J_\nu(t\sqrt{s})$ in a series of these functions,

$$s^{-\frac{\nu}{2}} J_\nu(t\sqrt{s}) = \sum a_n F_n(\nu+1, \nu+1, s),$$

and obtain

$$a_n = (2n+\nu+1) \left[\frac{\Gamma(n+\nu+1)}{n! \Gamma(\nu+1)} \right]^2 \int_0^1 s^{\frac{\nu}{2}} J_\nu(t\sqrt{s}) F_n(\nu+1, \nu+1, s) ds.$$

For the computation of the remaining integral, we use the Weber-Schafheitlin formula*,

* G. N. Watson, *Theory of Bessel Functions* (Cambridge; at the University Press, 1944) p. 401.

$$\int_0^\infty J_{2n+\nu+1}(t)J_\nu(st) dt = \begin{cases} \frac{s^\nu \Gamma(n+\nu+1)}{n! \Gamma(\nu+1)} F_n(\nu+1, \nu+1, s^2); & s < 1 \\ 0 & ; s > 1, \end{cases}$$

and obtain by the Hankel inversion formula

$$\frac{\Gamma(n+\nu+1)}{n! \Gamma(\nu+1)} \int_0^1 s^{\nu+1} F_n(\nu+1, \nu+1, s^2) J_\nu(ts) ds = \frac{J_{2n+\nu+1}(t)}{t}.$$

This we can write in the form

$$\frac{\Gamma(n+\nu+1)}{n! \Gamma(\nu+1)} \int_0^1 s^{\frac{\nu}{2}} J_\nu(t\sqrt{s}) F_n(\nu+1, \nu+1, s) ds = 2 \frac{J_{2n+\nu+1}(t)}{t}.$$

Thus we get

$$J_\nu(t\sqrt{s}) = \sum \frac{(4n+2\nu+2)\Gamma(n+\nu+1)}{n! \Gamma(\nu+1)} s^{\frac{\nu}{2}} F_n(\nu+1, \nu+1, s) \frac{J_{2n+\nu+1}(t)}{t}$$

and find from the orthogonality of $F_n(\nu+1, \nu+1, s)$

$$\int_0^1 s J_\nu(ts) J_\nu(us) ds = \frac{1}{2} \int_0^1 J_\nu(t\sqrt{s}) J_\nu(u\sqrt{s}) ds = \sum (4n+2\nu+2) \frac{J_{2n+\nu+1}(t) J_{2n+\nu+1}(u)}{ut}.$$

Furthermore our here developed theory guarantees the orthogonality of the functions

$$\frac{J_{2n+\nu+1}(t)}{t}$$

in the region $0 \leq t \leq \infty$ with the density function t .

We have

$$\int_0^\infty J_{2n+\nu+1}(t) J_{2m+\nu+1}(t) \frac{dt}{t} = \frac{\delta_{nm}}{4n+2\nu+2}.$$

This is another form of the Weber-Schafheitlin integral.

Therefore it is clear that the equation

$$f(t) = \int_0^\infty u f(u) \int_0^1 s J_\nu(ts) J_\nu(us) ds$$

has the solutions

$$f(t) = \frac{J_{2n+\nu+1}(t)}{t}; \quad n=0, 1, 2, \dots,$$

and we can expect that any solution of the equation can be expanded in the form

$$f(t) = \sum \frac{J_{2n+\nu+1}(t)}{t} (4n+2\nu+2) \int_0^\infty f(u) J_{2n+\nu+1}(u) du$$