

Directed Scale-Free Transport Networks Induced by a Rank-Based Routing Strategy

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We present a transport network model based on complex networks. In this model, the transport vector from one node to its nearest neighbor explicitly relies on the rank of the target's nearest neighbor according to the node potential measure, and the probability to link the target node is its ranking power-law function with exponent α . We find that the transport networks based on scale-free networks are scale-free and the power-law exponent γ is independent of α , and the same holds for the substrate network. The transport network based on a random network also has a scale-free degree distribution, but the power-law exponent γ depends on the exponent α . By the curve fitting method, the relation between γ and α is obtained.

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I. INTRODUCTION

Complex networks as an indispensable tool for the description of different complex systems have recently attracted enormous attention [1–3], natural and social systems as well as technological webs have been successfully described through scale-free networks [4–6], which are characterized by a degree distribution with a broad tail $p(k) \sim k^{-\gamma}$ for large values of the degree. The growth and preferential attachment have inspired the introduction of the Barabasi-Albert (BA) model [4], which led for the first time to a scale-free network with a power-law degree distribution with the exponent $\gamma = 3$. Krapivsky *et al.* [7] have demonstrated that the scale-free nature of the network is destroyed for nonlinear preferential attachment, the only case in which the topology of the network is scale-free is the one where the preferential attachment is asymptotically linear. Later, there were a number of models which introduced the generation for the scale-free networks, such as the copying models [8], fitness-based models [9], optimization models [10], etc.

Often the transport process is caused by the local gradient of some transport entities, such as a chemical potential, temperature, force, material good, power, etc. To be able to consider the transport process in the network, it is reasonable to hypothesize the existence of a gradient field that governs the information flow. Based on this idea, Toroczkai and Bassler [11, 12] have put forward the complex gradient network, which introduces a different mechanism generating power-law degree distributed networks. They have found that the

distributions of gradient networks on scale-free networks obey a power-law distribution and the exponent γ is the same as that of the substrate networks. An inspiring result is that in the natural scaling limit the gradient networks on random graphs becomes scale-free with a connectivity exponent of $\gamma = 1$.

However, many investigations have shown that the exponents γ for a large number of scale-free networks take values between 2 and 3. In order to generate a wider range of the power exponent in random gradient networks, we systematically build a complex transport network model induced by a rank-based routing strategy, where the transport vector from one node to its nearest neighbor explicitly relies on the rank of the target nearest neighbor according to the node potential measure, and the probability to link the target node is its ranking power-law function with exponent α . We study the degree distributions of transport networks based on random and scale-free networks. The degree distribution of the transport network based on a scale-free network is also scale-free, the power-law exponent γ is independent of α , and the same with the substrate network. If the substrate network is random, the degree distribution is also a power-law; by the curve fitting method the relation between γ and α can be described as $\gamma = 1 + 1/\alpha$. In other words, our complex transport network model can show the degree distribution with a wider range of the power exponent.

II. MODEL

The gradient networks describe the gradient-based transport mode. In fact, the information is transported with the maximum probability in this way in many systems. Based on this idea, we introduce the complex transport network model which is induced by the rank-based routing strategy. The model can be defined as follows: consider a substrate network with N nodes, the substrate network can be random or scale-free. Associated with each node i , there is a scalar h_i that describes the ‘‘potential’’ of the node. Then elicit a directed line from every node i , and the end point of this directed line is selected to follow the principle: if the node i has the minimum potential of all its neighbors, then this directed line points at itself and is formed into a self-loop; if there are m neighbors with less potential compared with i (we use j to denote the m nodes, $j=1, 2, \dots, m$), then rank these nodes according to their potential and the ranking of j is marked by R_j . It is noted that the rank of the node with the minimum potential is 1, while the rank of the node with the maximum potential is m , that is to say $1 \leq R_j \leq m$. The probability of the directed line pointing from i to j is considered as

$$P(i \rightarrow j) = \frac{R_j^{-\alpha}}{\sum_{k=1}^m R_k^{-\alpha}}, \quad (1)$$

where α is a positive control parameter. It is obvious that the connective probability for the two nodes is decreasing with increasing R_m . After regulating α from 0 to ∞ , we can obtain all kinds of transport, from random transport to gradient transport. When α tends to ∞ ,

the probability that the directed out-line points to the node with the minimum potential approaches 1; in this case our model returns to the gradient network model [10]. On the other hand, it corresponds to a random transport situation when $\alpha = 0$.

III. NUMERICAL SIMULATIONS

In this section, we investigate the degree distributions in order to characterize the topological structure of the transport networks.

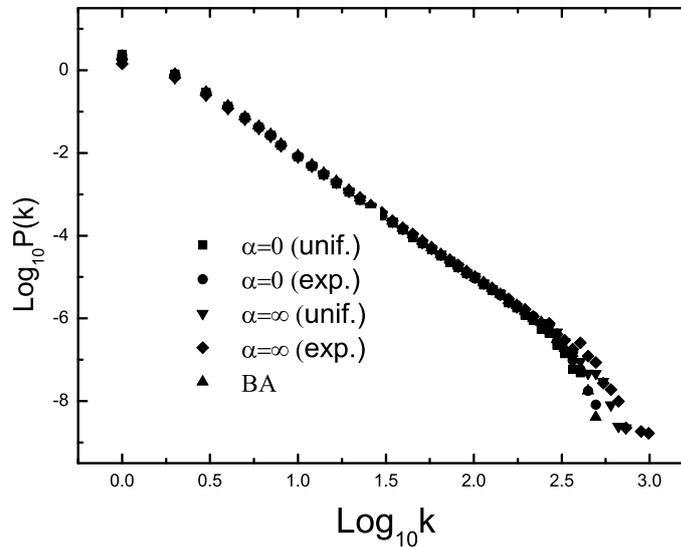


FIG. 1: The degree distributions $p(k)$ for different conditions of α and scalar field distribution in transport networks based on the scale-free networks; the degree distribution of the BA model has also been shown.

First, we use scale-free networks as the substrate networks to study the topological properties of the transport networks. In this condition, the BA model is adopted. The total size of the network is $N = 50000$, and an all ensemble average is computed over a set of 1000 network realizations. The different distribution forms of the scalar field are considered.

In references [11, 12] the authors have proved that the gradient networks on scale-free networks are also scale-free and are characterized by the same power-law exponent with the associated substrate networks. For random transport networks, the directed line pointing from i to its neighbors j is chosen randomly. That is, for each node, the number of its incoming link is in direct proportion to its degree. So that, regardless of substrate

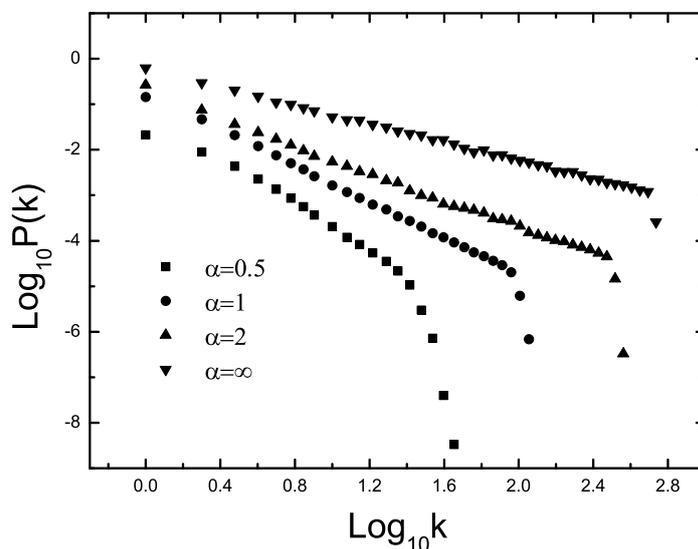


FIG. 2: The degree distributions $p(k)$ for different values of α in transport networks based on the random networks, where h is the uniform distribution.

network topology, the random transport networks have the same degree distributions with the associated substrate networks. Since the gradient networks and random transport networks based on scale-free networks have the same degree distribution, it is reasonable that we suppose that all transport networks based on scale-free networks induced by the rank-based routing strategy should have the same degree distribution.

In addition, our transport network model is induced by the rank-based routing strategy. That is, for each node i , the probability of the directed line pointing from i to its neighbors j is determined by the rank of $h(j)$. The rank of $h(j)$ characterizes its relative value in all the neighbors of i , not its absolute value. The absolute value of $h(j)$ is relevant to the distribution of the scalar field, however the relative rank of $h(j)$ is irrelevant to the distribution of the scalar field. So we can suppose that, regardless of substrate network topology, the degree distributions of the transport networks is irrelevant to the distribution of the scalar field $h(i)$.

Figure 1 provides the degree distributions of transport networks in different conditions of α and scalar field distributions, and the degree distribution of the substrate network has also been shown. We have found that the curves overlap with each other and display power-law distributions for all α . It can be said that the degree distributions are not only uncorrelated with α but also independent of the distributions of the scalar field. These findings have confirmed our above suppositions.

In the following we have used random networks as the substrate networks to study

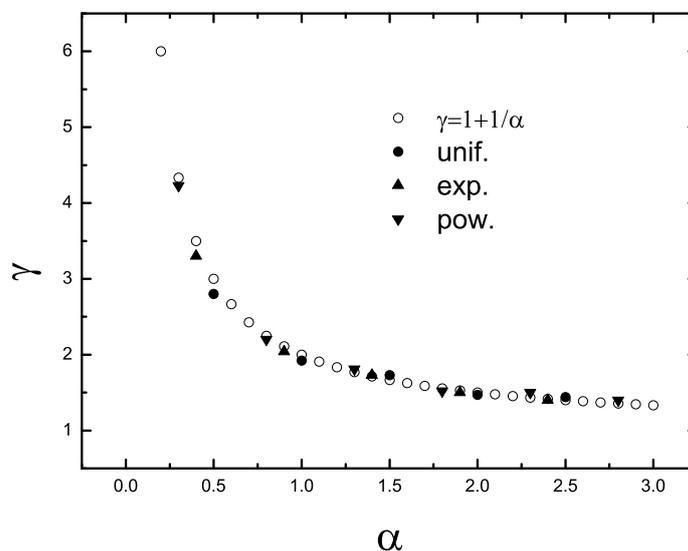


FIG. 3: The relation between the degree distribution exponent γ and α in transport networks with different distributions of the scalar field: uniform distribution, exponent distribution, and power-law distribution. As a comparative function, the curve for $\gamma = 1 + 1/\alpha$ has also been drawn. The substrate network is the random network.

the degree distributions of transport networks. We use the ER model [1] to generate a random network where each pair of nodes is linked with probability $p = \frac{\langle k \rangle}{N}$. The network size is $N = 1000$ and the connectivity probability is $p = 0.1$. The statistical average is performed over 1000 samples.

In references [11, 12], the authors have proved that the gradient network on the ER network has a power-law degree distribution with exponent $\gamma = 1$. Since the random transport networks have the same degree distributions with the associated substrate networks, the random transport networks on random works should be random. For two kinds of special cases of random transport ($\alpha = 0$) and gradient networks ($\alpha = \infty$), they have different distributions; it is reasonable that we suppose that all transport networks induced by the rank-based routing strategy should have different degree distributions if the associated substrate networks are random. Our suppositions have been confirmed by Figure 2. In Figure 2, the degree distributions for different values of α have been shown. We have found that for all simulated $\alpha > 0$, the degree distributions of transport networks based on random networks are also scale-free. However, the power-law exponent γ is increasing with increasing α .

Figure 3 gives the relation between α and the degree distribution exponent γ of the transport networks under different distribution conditions of the scalar field: uniform

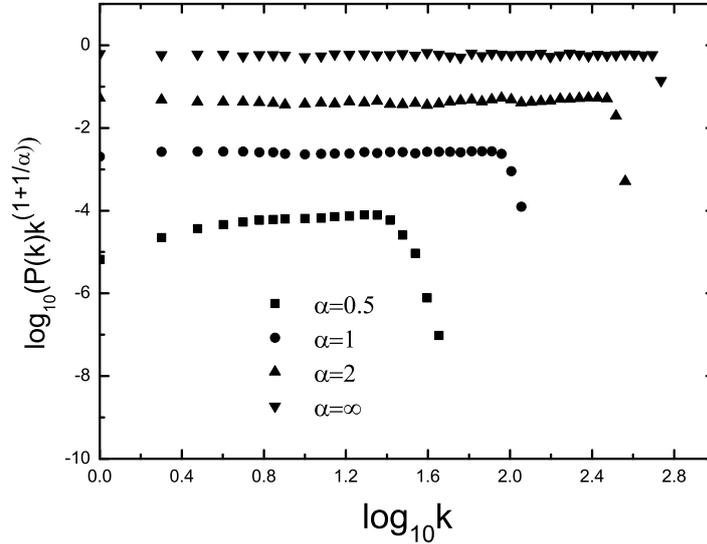


FIG. 4: The relation between $P(k)k^{1+1/\alpha}$ and k with different values of α . The substrate network is the random network.

distribution, exponent distribution, and power-law distribution. We also find that the exponent α is uncorrelated with the distribution of the scalar field, which has confirmed our corresponding suppositions. By the curve fitting method, we have discovered that all points are located on the curve $\gamma = 1 + 1/\alpha$ within the error range.

In the following, we further verify that the degree distribution exponent γ satisfies the equation $\gamma = 1 + 1/\alpha$. Suppose the relationship is established, then

$$P(k) = ak^{-(1+1/\alpha)}, \quad P(k)k^{1+1/\alpha} = a, \quad (2)$$

where a is an uncorrelated constant with degree k .

In Figure 4, we have shown the relation between $P(k)k^{1+1/\alpha}$ and k with different values of α , which verifies the guess that the variable $P(k)k^{1+1/\alpha}$ is independent of the degree k without regarding the effect in a certain scaling limit, it further validates the relation between γ and α through curve fitting.

IV. CONCLUSIONS

We have proposed a model of a complex transport network which is induced by rank-based routing. After investigating the degree distributions of transport networks which are based on the substrate random and scale-free networks, we have discovered that the degree

distributions are uncorrelated with the distributions of the scalar field. If the substrate network is scale-free, the degree distributions of transport networks are all scale-free, and the power-law exponent is the same as with the substrate scale-free network but independent of α . If the substrate network is random, the degree distributions are also power-laws, the power-law exponent γ is increasing with increasing α . By the curve fitting method, the relation between γ and α can be described as $\gamma = 1 + 1/\alpha$. So this transport network model shows the degree distribution with a wider range of the power exponent.

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