Schwinger Method and Path Integral with Generalized Canonical Transformation for a Harmonic Oscillator with Time-Dependent Mass and Frequency

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The exact propagator for a harmonic oscillator with time-dependent mass and frequency is found by the Schwinger method and a path integral with a generalized canonical transformation. In the Schwinger formalism, the propagator can be obtained by basic operator algebra and elementary integrations. In the path integral method, it can be shown that such a propagator can be derived from that for a unit mass and frequency oscillator in a new space-time coordinate system with the help of a generalized canonical transformation. The power of propagator methods for solving time-dependent Hamiltonian systems is also discussed.

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I. INTRODUCTION

In recent years, the study of how to find quantum solutions for time-dependent Hamiltonian systems has become quite interesting [1-6]. Pepore *et al.* [7] applied the path integral method to calculate the propagator for a harmonic oscillator with a time-dependent mass and frequency, and then expanded the obtained propagator to derive the time-dependent wave function. The various applications in many areas of physics, such as quantum optics, [8] cosmology, [9] and nanotechnology [10] are the main reasons for these intensive studies in the physics literature. The aim of this paper is to derive the propagator for a harmonic oscillator with time-dependent mass and frequency by two methods. The first method is the Schwinger method [11], while the second method is the path integral with a generalized canonical transformation [12].

Recently, there has been a report on the calculating of non-relativistic propagators by the Schwinger method [13–17]. Aragao *et al.* [13] applied this method to derive the propagator for a charged particle in a uniform magnetic field in 2007. In 2003, Barone *et al.* [14] evaluated the propagator for a simple harmonic oscillator and compared it with the path integral method. In this paper we will generalize their results [13–17] to the case of time-dependent Hamiltonian systems.

Quantum canonical transformations [18–20] that include both point transformations and time evolution are adequate applications for transforming a system of interest into a

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simple equation whose solution is known. Bouloudenine and *et al.* [18–19] used quantum canonical transformations in connection with the Schwinger method to calculate the propagator for time-dependent quadratic systems both, 1 and 3 dimensional. In this paper we will apply the quantum canonical transformations with the path integral method to calculate the propagator for a harmonic oscillator with time-dependent mass and frequency.

In Section II, the procedures of the Schwinger method are described. In Section III, the propagator for a harmonic oscillator with a time-dependent mass and frequency are derived by Schwinger's method. The path integral with generalized canonical transformation is applied to calculate the propagator for a harmonic oscillator with time-dependent mass and frequency in Section IV. Finally, the conclusion is given in Section V.

II. PROCEDURES OF THE SCHWINGER METHOD

Begin by considering a time-dependent Hamiltonian $H(t)$; the propagator is defined by [21]

$$K(x,x';\tau) = \langle x|\hat{T}\exp\left(-\frac{i}{\hbar}\int_0^\tau \hat{H}(t)dt\right)|x'\rangle, \tag{1}$$

where $\hat{T}$ is the time-ordering operator and $|x\rangle$, $|x'\rangle$ are the eigenvectors of the position operator $\hat{x}$ (in the Schrödinger picture) with eigenvalues x and x', respectively.

The differential equation for the propagator in Eq. (1) can be written as

$$i\hbar\frac{\partial}{\partial\tau}K(x,x';\tau) = \langle x|\hat{H}\exp\left(-\frac{i}{\hbar}\int_0^\tau \hat{H}(t)dt\right)|x'\rangle. \tag{2}$$

Using the relation between operators in the Heisenberg and Schrödinger pictures, we get the equation for the propagator in the Heisenberg picture:

$$i\hbar\frac{\partial}{\partial\tau}K(x,x';\tau) = \langle x(\tau)|\hat{H}(\hat{x}(\tau),\hat{p}(\tau))|x'(0)\rangle, \tag{3}$$

where $|x(\tau)\rangle$ and $|x'(0)\rangle$ are the eigenvectors of the operators $\hat{x}(\tau)$and $\hat{x}(0)$, respectively, with the corresponding eigenvalues $x$and x'. Besides, $\hat{x}(\tau)$ and $\hat{p}(\tau)$ satisfy the Heisenberg equations:

$$i\hbar\frac{d\hat{x}(\tau)}{d\tau} = [\hat{x}(\tau),\hat{H}], \qquad i\hbar\frac{d\hat{p}(\tau)}{d\tau} = [\hat{p}(\tau),\hat{H}]. \tag{4}$$

The main idea of the Schwinger method consists in the following steps.

(1). The first step is solving the Heisenberg equations for $\hat{x}(\tau)$ and $\hat{p}(\tau)$, and writing the solution for $\hat{p}(\tau)$ only in terms of the operators $\hat{x}(\tau)$ and $\hat{x}(0)$.

(2). Next, we substitute the solutions obtained in step (1) into the expression for $\hat{H}(\hat{x}(\tau),\hat{p}(\tau))$ in Eq. (3) and use the commutator $[\hat{x}(0),\hat{x}(\tau)]$ to rewrite each term of $\hat{H}$ in

a time ordered form with all operators $\hat{x}(\tau)$ to the left and all operators $\hat{x}(0)$ to the right. The time ordered Hamiltonian can be defined as $\hat{H}_{ord}(\hat{x}(\tau), \hat{x}(0))$.

(3). After this ordering, Eq. (3) can be written in the form

$$i\hbar \frac{\partial}{\partial \tau} K(x, x'; \tau) = H(x, x'; \tau) K(x, x'; \tau), \tag{5}$$

with $H(x, x'; \tau)$ being an ordinary function defined as

$$H(x, x'; \tau) = \frac{\langle x(\tau) | \hat{H}_{ord}(\hat{x}(\tau), \hat{x}(0) | x'(0) \rangle}{\langle x(\tau) | x'(0) \rangle}. \tag{6}$$

Integrating Eq. (5) over τ, the propagator takes the form

$$K(x, x'; \tau) = C(x, x') \exp\left\{ -\frac{i}{\hbar} \int_0^\tau H(x, x'; \tau') dt \right\}, \tag{7}$$

where $C(x, x')$ is an integration constant.

(4). The last step is the calculation of $C(x, x')$. This is obtained by using the following conditions:

$$-i\hbar \frac{\partial}{\partial x} K(x, x'; \tau) = \langle x(\tau) | \hat{p}(\tau) | x'(0) \rangle, \tag{8}$$

$$i\hbar \frac{\partial}{\partial x'} K(x, x'; \tau) = \langle x(\tau) | \hat{p}(0) | x'(0) \rangle, \tag{9}$$

and the initial condition

$$\lim_{\tau \to 0^+} K(x, x'; \tau) = \delta(x - x'). \tag{10}$$

III. SCHWINGER METHOD FOR A HARMONIC OSCILLATOR WITH TIME-DEPENDENT MASS AND FREQUENCY

The Hamiltonian for a harmonic oscillator with time-dependent mass and frequency can be written as [7]

$$H(\tau) = \frac{p^2}{2m(\tau)} + \frac{1}{2} m(\tau) \omega^2(\tau) x^2, \tag{11}$$

where $m(\tau)$ and $\omega(\tau)$ are the time-dependent mass and frequency, respectively.

The equation of motion corresponding to the Hamiltonian in Eq. (11) is

$$\ddot{x} + \left(\frac{\dot{m}(\tau)}{m(\tau)} \right) \dot{x} + \omega^2(\tau) x = 0. \tag{12}$$

The solution of Eq. (12) can be written in the form

$$x(\tau) = \alpha(\tau)\left\{A\cos[\gamma(\tau)] + B\sin[\gamma(\tau)]\right\}, \tag{13}$$

where $\alpha(\tau) = \frac{s(\tau)}{\eta(\tau)}$ and $\eta(\tau) = \sqrt{m(\tau)}$.

The functions $\alpha(\tau)$ and $\gamma(\tau)$ can be determined by substituting Eq. (13) into Eq. (12):

$$\ddot{s} - \dot{\gamma}^2 s + \left[\omega^2 - \frac{\ddot{\eta}}{\eta}\right] s = 0, \tag{14}$$

$$\dot{\gamma} s^2 = C^2, \tag{15}$$

where C is a constant independent of the initial conditions.

The constants A and B in Eq. (13) can be determined by imposing the initial conditions $x(0) = x'$ and $p(0) = p'$. After some algebraic manipulations it can be shown that

$$x(\tau) = x' f(\tau) + p' g(\tau), \tag{16}$$

where

$$f(\tau) = \alpha(\tau)\left\{a\cos\gamma(\tau) + c\sin\gamma(\tau)\right\}, \tag{17}$$

$$g(\tau) = \alpha(\tau)\left\{b\cos\gamma(\tau) + d\sin\gamma(\tau)\right\}, \tag{18}$$

with $a, b, c,$ and d given by

$$a = \left[\frac{\eta'}{s'}\cos\gamma' + \frac{(\eta'\dot{s}' - s'\dot{\eta}')}{C^2}\sin\gamma'\right], \tag{19}$$

$$b = -\frac{s'\sin\gamma'}{\eta' C^2}, \tag{20}$$

$$c = \left[\frac{\eta'}{s'}\sin\gamma' - \frac{(\eta'\dot{s}' - s'\dot{\eta}')}{C^2}\cos\gamma'\right], \tag{21}$$

$$d = \frac{s'\cos\gamma'}{\eta' C^2}. \tag{22}$$

By solving the Heisenberg equation in Eq. (4), the position operator $\hat{x}(\tau)$ can be written similarly to Eq. (16) as

$$\hat{x}(\tau) = \hat{x}(0) f(\tau) + \hat{p}(0) g(\tau). \tag{23}$$

The momentum operator $\hat{p}(\tau) = m(\tau)\dot{\hat{x}}(\tau)$ can be written by using Eq. (23) as

$$\hat{p}(\tau) = \eta^2(\tau)\hat{x}(0)\dot{f}(\tau) + \eta^2(\tau)\hat{p}(0)\dot{g}(\tau). \tag{24}$$

The next step is expressing $\hat{p}(\tau)$ only in terms of $\hat{x}(\tau)$ and $\hat{x}(0)$ by eliminating $\hat{p}(0)$ from Eq. (24) with the helping of Eq. (23), the result is

$$\hat{p}(\tau) = \eta^2(\tau)\hat{x}(0)g(\tau)\left[\frac{f(\tau)}{g(\tau)}\right]^{\circ} + \eta^2(\tau)\hat{x}(\tau)\frac{\dot{g}(\tau)}{g(\tau)}. \tag{25}$$

where $\left[\frac{f(\tau)}{g(\tau)}\right]^{\circ}$ denotes differentiating with respect to time.

Substituting $\hat{x}(\tau)$ and $\hat{p}(\tau)$ into the Hamiltonian operator $\hat{H}(\tau) = \frac{\hat{p}^2(\tau)}{2\eta^2(\tau)} + \frac{1}{2}\eta^2(\tau)\omega^2(\tau)\hat{x}^2(\tau)$, the result is

$$\hat{H}(\tau) = \frac{\eta^2(\tau)}{2}\left[\begin{array}{l} x^2(\tau)\left\{\left[\frac{\dot{g}(\tau)}{g(\tau)}\right]^2 + \omega^2(\tau)\right\} + \hat{x}^2(0)g^2(\tau)\left\{\left[\frac{f(\tau)}{g(\tau)}\right]^{\circ}\right\}^2 \\ +2\hat{x}(\tau)\hat{x}(0)\dot{g}(\tau)\left[\frac{f(\tau)}{g(\tau)}\right]^{\circ} + [\hat{x}(0),\hat{x}(\tau)]\dot{g}(\tau)\left[\frac{f(\tau)}{g(\tau)}\right]^{\circ} \end{array}\right], \tag{26}$$

where this Hamiltonian has been written in an appropriate operator form by adding and subtracting

$$\hat{x}(\tau)\hat{x}(0)\dot{g}(\tau)\left[\frac{f(\tau)}{g(\tau)}\right]^{\circ}.$$

Applying the commutator $[\hat{x}(0),\hat{x}(\tau)] = i\hbar g(\tau)$ to Eq. (26) and using Eqs. (5)–(7), the propagator takes the form

$$\begin{aligned} K(x,x';\tau) &= C(x,x')\exp\left(-\tfrac{i}{\hbar}\int_0^{\tau}\tfrac{\langle x(t)|\hat{H}_{ord}|x'(0)\rangle}{\langle x(t)|x'(0)\rangle}dt\right) \\ &= C(x,x')\exp\Bigg[-\tfrac{i}{\hbar}\int_0^{\tau}\Big\{\tfrac{\eta^2(t)}{2}x^2[\tfrac{\dot{g}^2(t)}{g^2(t)} + \omega^2(t)] + \tfrac{\eta^2(t)}{2}x'^2g^2(t)\left[\tfrac{f(t)}{g(t)}\right]^{\circ 2} \\ &+\eta^2(t)xx'\dot{g}(t)\left[\tfrac{f(t)}{g(t)}\right]^{\circ} + \tfrac{i\hbar}{2}\eta^2(t)g(t)\dot{g}(t)\left[\tfrac{f(t)}{g(t)}\right]^{\circ}\Big\}dt\Bigg]. \end{aligned} \tag{27}$$

Now, we will integrate each term of Eq. (27) with respect to time. Beginning by modifying the functions $f(t)$ and $g(t)$ in Eqs. (17) and (18), the result is

$$f(t) = \alpha(t)\left\{\frac{\eta'}{s'}\cos[\gamma(t)-\gamma'] - \frac{(\eta'\dot{s}' - s'\dot{\eta}')}{C^2}\sin[\gamma(t)-\gamma']\right\}, \tag{28}$$

$$g(t) = \frac{\alpha s'}{\eta' C^2}\sin[\gamma(t)-\gamma']. \tag{29}$$

Using Eq. (29) to derive $\frac{\dot{g}^2(t)}{g^2(t)}$, it can be obtained that

$$\begin{aligned}\frac{\dot{g}^2(t)}{g^2(t)} &= \dot{\gamma}^2(t)\cot^2[\gamma(t)-\gamma'] + \frac{2\dot{\gamma}(t)[\eta(t)\dot{s}(t)-s(t)\dot{\eta}(t)]}{\alpha(t)\eta^2(t)}\cot[\gamma(t)-\gamma'] \\ &+ \frac{(\eta(t)\dot{s}(t)-s(t)\dot{\eta}(t))^2}{\alpha^2(t)\eta^4(t)}.\end{aligned} \tag{30}$$

¿From Eq. (14), $\omega^2(t)$ can be rewritten as

$$\omega^2(t) = \dot{\gamma}^2(t) + \frac{\ddot{\eta}(t)}{\eta(t)} - \frac{\ddot{s}(t)}{s(t)}. \tag{31}$$

Substituting Eqs. (30) and (31) into the first term of Eq. (27) or $-\frac{ix^2}{2\hbar}\int\limits_0^\tau \eta^2(t)\left[\frac{\dot{g}^2(t)}{g^2(t)}+\omega^2(t)\right]dt$, the result is

$$-\frac{ix^2}{2\hbar}\int\limits_0^\tau \eta^2(t)\left[\frac{\dot{g}^2(t)}{g^2(t)}+\omega^2(t)\right]dt$$

$$\begin{aligned}&= \frac{ix^2}{2\hbar}\int\limits_0^\tau [-\eta^2(t)\dot{\gamma}^2\csc^2[\gamma(t)-\gamma'] - \frac{2\eta(t)\gamma(t)(\eta\dot{s}-s\dot{\eta})}{s(t)}\cot[\gamma-\gamma'] \\ &-\eta(t)\ddot{\eta}(t) + \frac{\eta^2(t)\ddot{s}(t)}{s(t)} - \frac{(\eta(t)\dot{s}(t)-s(t)\dot{\eta}(t))^2}{s^2(t)}]dt\end{aligned}$$

$$\begin{aligned}&= \frac{ix^2}{2\hbar}\int\limits_0^\tau \frac{d}{dt}\left\{\eta^2(t)\dot{\gamma}(t)\cot[\gamma(t)-\gamma'] + \frac{\eta^2(t)(\dot{s}(t)\eta(t)-s(t)\dot{\eta}(t))}{\eta(t)s(t)}\right\}dt \\ &= \frac{i}{2\hbar}mx^2\left(\frac{\dot{s}\eta-s\dot{\eta}}{s\eta}\right) + \frac{i}{2\hbar}\dot{\gamma}mx^2\cot(\gamma-\gamma').\end{aligned} \tag{32}$$

The next step is calculating the second term of Eq. (27) by the help of the identity

$$\left[\frac{f(t)}{g(t)}\right]^\circ = -\frac{C^2\eta'^2\dot{\gamma}(t)}{s'^2}\csc^2[\gamma(t)-\gamma']. \tag{33}$$

The second term of Eq. (27) can be written as

$$-\frac{ix'^2}{2\hbar}\int\limits_0^\tau \eta^2(t)g^2(t)\left\{\left[\frac{f(t)}{g(t)}\right]^\circ\right\}^2 dt = -\frac{i}{2\hbar}\eta'^2x'^2\dot{\gamma}'\int\limits_0^\tau \dot{\gamma}(t)\csc^2[\gamma-\gamma']dt$$

$$= -\frac{i}{2\hbar}\dot{\gamma}'m'x'^2\cot(\gamma-\gamma'). \tag{34}$$

The third term of Eq. (27) can be calculated with the help of Eqs. (29) and (33) as

$$-\frac{i}{\hbar}xx'\int\limits_0^\tau \eta^2(t)\dot{g}(t)\left[\frac{f(t)}{g(t)}\right]^\circ dt = \frac{ixx'\eta'}{s}\int\limits_0^\tau \{(\eta(t)\dot{s}(t)-s(t)\dot{\eta}(t))\,\dot{\gamma}(t)\csc[\gamma(t)-\gamma']$$

$$+\eta(t)\dot{\gamma}^2(t)s(t)\cot[\gamma(t)-\gamma']\csc[\gamma(t)-\gamma']\}dt$$

$$= -\frac{i}{\hbar}\eta\eta'\sqrt{\dot{\gamma}\dot{\gamma}'}xx'\csc(\gamma-\gamma'), \tag{35}$$

where we have used the relation

$$\dot{\gamma}(t)\dot{s}(t) = -\frac{1}{2}\ddot{\gamma}(t)s(t) = -\frac{\ddot{\gamma}C}{2\sqrt{\dot{\gamma}}}.$$

The last term of Eq. (27) can be evaluated by

$$\frac{1}{2}\int_0^\tau \eta^2(t)g(t)\dot{g}(t)\left[\frac{f(t)}{g(t)}\right]^\circ dt = -\frac{1}{2}\int_0^\tau \left\{\frac{\dot{s}(t)}{s(t)} - \frac{\dot{\eta}(t)}{\eta(t)} + \dot{\gamma}(t)\cot[\gamma(t)-\gamma']\right\}dt$$

$$= -\frac{1}{2}\int_0^\tau \frac{d}{dt}\left(\ln[s's(t)] - \ln[\eta'\eta(t)] + \ln[\sin(\gamma-\gamma')]\right)dt$$

$$= -\frac{1}{2}\ln(C^2) + \frac{1}{2}\ln(\sqrt{\dot{\gamma}\dot{\gamma}'}) + \frac{1}{2}\ln(\eta'\eta) - \frac{1}{2}\ln[\sin(\gamma-\gamma')]. \tag{36}$$

Combining the result in Eq. (32), (34), (35), and (36), the propagator can be written as

$$K(x,x';\tau) = C(x,x')\left(\frac{\eta\eta'\sqrt{\dot{\gamma}\dot{\gamma}'}}{\sin(\gamma-\gamma')}\right)^{\frac{1}{2}}\exp\left[\frac{i}{2\hbar}mx^2\left(\frac{\dot{s}\eta - s\dot{\eta}}{s\eta}\right)\right]$$

$$\times\exp\left[\frac{i}{2\hbar\sin(\gamma-\gamma')}\left[\left(m\dot{\gamma}x^2 + m'\dot{\gamma}'x'^2\right)\cos(\gamma-\gamma') - 2\eta\eta'\sqrt{\dot{\gamma}\dot{\gamma}'}xx'\right]\right]. \tag{37}$$

The final step is finding the function $C(x,x')$. Substituting Eq. (37) into Eq. (3), it can be obtained that

$$i\hbar\frac{\partial C(x,x')}{\partial x'} = \frac{\eta'\left(\eta'\dot{s}' - s'\dot{\eta}'\right)}{s'}x'C(x,x'). \tag{38}$$

The solution of Eq. (38) can be written as

$$C(x,x') = C(x)\exp\left(-\frac{i}{2\hbar}m'x'^2\left(\frac{\eta'\dot{s}' - s'\dot{\eta}'}{\eta's'}\right)\right), \tag{39}$$

where $C(x)$is a position function.

The propagator in Eq. (37) can be expressed as

$$K(x,x';\tau) = C(x)\left(\frac{\eta\eta'\sqrt{\dot{\gamma}\dot{\gamma}'}}{\sin(\gamma-\gamma')}\right)^{\frac{1}{2}}\exp\left[\frac{i}{2\hbar}\left\{mx^2\left(\frac{\dot{s}\eta - s\dot{\eta}}{s\eta}\right) - m'x'^2\left(\frac{\dot{s}'\eta' - s'\dot{\eta}'}{s'\eta'}\right)\right\}\right]$$

$$\times \exp\left\{\frac{i}{2\hbar\sin(\gamma-\gamma')}\left[\left(m\dot{\gamma}x^2+m'\dot{\gamma}'x'^2\right)\cos(\gamma-\gamma')-2\eta\eta'\sqrt{\dot{\gamma}\dot{\gamma}'}xx'\right]\right\}. \tag{40}$$

The next step is calculating $C(x)$. Substituting Eq. (40) into Eq. (8), the result is

$$\frac{\partial C(x)}{\partial x}=0, \tag{41}$$

which implies that $C(x)$is a constant independent of x.

After applying Eq. (10), it can be obtained that

$$C(x)=\frac{1}{\sqrt{2\pi i\hbar}}. \tag{42}$$

So, the propagator for a harmonic oscillator with time-dependent mass and frequency can be written as

$$K(x,x';\tau)=\left(\frac{\eta\eta'\sqrt{\dot{\gamma}\dot{\gamma}'}}{2\pi i\hbar\sin(\gamma-\gamma')}\right)^{\frac{1}{2}}\exp\left\{\frac{i}{2\hbar}\left[mx^2\left(\frac{\dot{s}\eta-s\dot{\eta}}{s\eta}\right)-m'x'^2\left(\frac{\dot{s}'\eta'-s'\dot{\eta}'}{s'\eta'}\right)\right]\right\}$$

$$\times \exp\left\{\frac{i}{2\hbar\sin(\gamma-\gamma')}\left[\left(m\dot{\gamma}x^2+m'\dot{\gamma}'x'^2\right)\cos(\gamma-\gamma')-2\eta\eta'\sqrt{\dot{\gamma}\dot{\gamma}'}xx'\right]\right\}. \tag{43}$$

This propagator is the same as the result of Pepore *et al* [7], found by using the path integral method.

IV. PATH INTEGRAL WITH GENERALIZED CANONICAL TRANSFORMATION

Recently, Boudjema and Boudjedaa [18] used the Schwinger method in connection with a quantum canonical transformation to study the 3-dimensional time-dependent quadratic systems. In that work [18] the quantum canonical transformation was used to simplify the problem from a quadratic Hamiltonian to a free particle. In this section, we will use the Feynman path integral [21] in connection with a quantum canonical transformation to calculate the propagator for a harmonic oscillator with time-dependent mass and frequency.

In the canonical formulation of the path integral [21], the propagator can be written as

$$K(x'',t'';x',t')=\int DxDp\exp\left[\frac{i}{\hbar}\int_{t'}^{t''}(p\dot{x}-H)dt\right], \tag{44}$$

where the Hamiltonian is

$$H(t)=\frac{p^2}{2m(t)}+\frac{1}{2}m(t)\omega^2(t)x^2. \tag{45}$$

We try to find a transformation that can transform the system with the Hamiltonian in Eq. (45) into the harmonic oscillator with unit mass and frequency. The canonical and time transformation $(x, p, t) \to (Q, P, \tau)$ can be proposed as [12]

$$P = A(t)p, \tag{46}$$

$$Q = \frac{x}{A(t)}, \tag{47}$$

$$d\tau = \frac{dt}{m(t)A^2(t)}, \tag{48}$$

which has the generating function

$$F_2(x, P, t) = \frac{xP}{A(t)}, \tag{49}$$

where $A(t)$ is a function to be determined.

Using a process similar to Lawande and Dhara [22], we obtain the transformation of the measures as follows:

$$DxDp = \frac{1}{\sqrt{A'A''}} DQDP, \tag{50}$$

where $A' = A(t')$ and $A'' = A(t'')$. By making use of Eq. (46)–(48) and Eq. (50) the propagator can be rewritten as

$$K(x'', t''; x', t') = \frac{1}{\sqrt{A'A''}} \exp\left\{\frac{i}{2\hbar}[\frac{m''\dot{A}''x''^2}{A''} - \frac{m'\dot{A}'x'^2}{A'}]\right\}$$

$$\int DQDP \exp\left[\frac{i}{\hbar}\int_{\tau'}^{\tau''}(P\bar{Q} - G)d\tau\right], \tag{51}$$

where $G = \frac{P^2+Q^2}{2}$ and $\bar{Q} = \frac{dQ}{d\tau}$, and $A(t)$ satisfies the auxiliary equation [23]

$$\ddot{A}(t) + \frac{\dot{m}(t)}{m(t)}\dot{A}(t) + \omega^2(t)A(t) = \frac{1}{m^2(t)A^3(t)}. \tag{52}$$

The propagator in the new system can be calculated by [12]

$$\begin{aligned} &\int DQDP \exp\left[\frac{i}{\hbar}\int_{\tau'}^{\tau''}(P\bar{Q} - G)d\tau\right] \\ &= \frac{1}{(2\pi i\hbar \sin(\tau''-\tau'))^{\frac{1}{2}}} \exp\left\{\frac{i}{2\hbar\sin(\tau''-\tau')}\left[(Q''^2 + Q'^2)\cos(\tau''-\tau') - 2Q'Q''\right]\right\}. \end{aligned} \tag{53}$$

Substituting Eq. (53) into Eq. (51), the result is

$$\begin{aligned} K(x'',t'';x',t') &= \left(\frac{1}{2\pi i\hbar A'A''\sin(\tau''-\tau')}\right)^{\frac{1}{2}} \exp\left\{\frac{i}{2\hbar}\left\{\frac{m''\dot{A}''x''^2}{A''} - \frac{m'\dot{A}'x'^2}{A'}\right\}\right\} \\ &\exp\left\{\frac{i}{2\hbar\sin(\tau''-\tau')}\left[(Q''^2+Q'^2)\cos(\tau''-\tau') - 2Q'Q''\right]\right\}. \end{aligned} \tag{54}$$

The final step is rewriting Eq. (54) into the original variables as

$$\begin{aligned} K(x'',t'';x',t') &= \left(\frac{1}{2\pi i\hbar A'A''\sin(\tau''-\tau')}\right)^{\frac{1}{2}} \exp\left\{\frac{i}{2\hbar}\left[\frac{m''\dot{A}''x''^2}{A''} - \frac{m'\dot{A}'x'^2}{A'}\right]\right\} \\ &\exp\left\{\frac{i}{2\hbar\sin(\tau''-\tau')}\left[\left(\frac{x''^2}{A''^2}+\frac{x'^2}{A'^2}\right)\cos(\tau''-\tau') - \frac{2x'x'''}{A''A'}\right]\right\}. \end{aligned} \tag{55}$$

In order to compare with Eq. (43), let $\frac{s(t)}{\eta(t)} = A(t)$ and $\gamma(t) = \int_{t'}^{t''} \frac{1}{m(t)\rho^2(t)}dt$, then the results agrees with Eq. (43).

V. CONCLUSIONS

In this paper we have successfully calculated the exact propagator for a harmonic oscillator with time-dependent mass and frequency by the Schwinger method and the path integral with a generalized canonical transformation. The resulting propagator in Eq. (43) and Eq. (55) is the same as in the report of Pepore *et al.* [7].

The crucial step in the Schwinger formalism is to find the solution of the Heisenberg equation Eq. (23) and to write the Hamiltonian operator in an appropriate order by the help of the commutator $[\hat{x}(0), \hat{x}(\tau)]$. In the path integral with generalized canonical transformation, the transformations of coordinate, momentum, and time in Eqs. (46)–(48) are the important steps used in modifying the problem from a time-dependent harmonic oscillator to a unit mass and frequency oscillator.

The advantage of the Schwinger method in this paper is that it requires only fundamental operator algebra and some basic integration. The path integral method in this article has an advantage in the point that it can transform the complicated system into a simplified problem.

Finally, we have presented several techniques in calculating the quantum solutions for non-conservative systems. It is preferable to have many tools forattacking the problems in the field of time-dependent Hamiltonian systems in addition to the standard method of the Lewis-Riesenfeld invariant operator [1–5] or the unitary operator method [6]. It may be suggested that the methods in this article can be applied to new famous problems, such as a time-dependent linear potential and a charged harmonic oscillator in a time-dependent electromagnetic field.

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