Effects of Atomic Motion on the Entropies and Entanglement for a JC Model with a Mixed State Input

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(Received May 24, 2006)

In this article, we study the effects of the atomic motion and the field-mode structure on the entropies and entanglement of a generalized JC model. We have investigated different forms of the intensity dependent function on the evolution of the entropies and entanglement in the case of a coherent superposition state and a statistical mixture of coherent states as the initial field states. We investigate the partial entropies of the atom and field subsystems numerically. The setting of the initial state of the field-mode and the atomic motion plays an important role in the evolution of the sub-entropies and entanglement.

PACS numbers: 42.50.-p, 42.50.Dv, 03.67.-a

I. INTRODUCTION

The Jaynes-Cummings (JC) model [1] has been generalized or extended further to incorporate the effects of the atomic motion and field mode structure [2, 3]. The influence of the atomic motion and field-mode structure on the atomic dynamics has been investigated [2]. It has been shown that the atomic motion and field mode structure give rise to nonlinear transient effects in the atomic population, which are similar to self-induced transparency and adiabatic effects. Quantum entanglement is widely used in quantum information processing [4]. In such physical processes one usually needs to find the entanglement properties and a way to control them. Therefore studying the dynamic properties of entanglement is useful for processing quantum information. Naturally it has become a subject of intensive study in fundamental physics. Many authors [5–15] have studied the evolution of the entropies as measures of entanglement for a non-decaying electromagnetic field mode interacting with a two-level atom in the JC model with the atom considered at rest for a superposition (SS) [16, 20] of coherent states. Whereas [21–24] are studies of the evolution of the entropies as a measure of entanglement while considering the initial field state to be a statistical mixture (SM) [19, 20] and the atom is considered to be at rest. The evolution of the entropies and entanglement have been studied by [25, 26] with a pure state of the system considering the effects of the atomic motion and field mode structure.

In this paper, we investigate the influences of the atomic motion, the field-mode structure, and the nonlinear intensity-dependent atom-field coupling $f(\hat{n})$ on the entropies and entanglement for a JC model with a supposed mixture states input. The material in this article is arranged as follows. In section II we introduce the model, present the reduced density operator of the field, which includes the effects of the atomic motion and the field-

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mode structure, and obtain an analytical solution. Using a numerical computation, we investigate the sub-entropies S_a, S_f and the particle-field entanglement by comparing the dynamics of the SS and SM states for one-photon input processes in section III. Finally, the conclusion is provided in section IV.

II. QUANTUM DYNAMICS OF A NONLINEAR JC MODEL WITH ATOMIC MOTION

The system considered here consists of a two-level particle (atom or trapped ion) in motion interacting with a single-mode quantized field via the one-quanta transition process. The effective Hamiltonian in the rotating-wave approximation is given by [3, 27]

$$\hat{\boldsymbol{H}} = \frac{\omega_0}{2}\hat{\sigma}_z + \omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \lambda g(\hat{t})[\hat{a}f(\hat{n})\sigma_+ + f(\hat{n})\hat{a}^+\sigma_-]\,, \tag{1}$$

where a^+ and a are the creation and annihilation operators of the field mode of frequency ω, $\hat{\sigma}_z$ is the population inversion operator, and $\hat{\sigma}_\pm$ are the "spin flip" operators, ω_0 is the transition frequency between the excited and ground states of the particle, and $\lambda f(\hat{n})$ represents an intensity-dependent atom-field coupling. In this article we consider certain forms for the function $f(\hat{n})$, $(\hat{I}, \sqrt{\hat{n}}, \sqrt{\frac{1}{\hat{n}}})$. $g(t)$ is the shape function of the cavity field mode. The atomic motion may be described by

$$g(t) = \cos\left(\frac{p\pi\nu t}{L}\right)\,, \tag{2}$$

where ν is the atomic motion velocity and p is a positive integer representing the number of half wavelengths of the field mode inside a cavity of length L. We consider the resonant case (i.e. $\omega = \omega_\circ$) and assume that the atom enters the cavity at time $t = 0$ in the excited state $|e\rangle$, i.e., $\hat{\rho}_a = |e\rangle\langle e|$, and leaves the cavity again after passing p half-wavelengths of the electric field. Furthermore the field is assumed to be initially in a superposition state (SS), i.e.,

$$\hat{\rho}^f_{SS}(0) = \frac{1}{A}[|\alpha\rangle\langle\alpha| + r(|\alpha\rangle\langle-\alpha| + |-\alpha\rangle\langle\alpha|) + r^2(|-\alpha\rangle\langle-\alpha|)]\,, \tag{3}$$

where $A = [1 + r^2 + 2r\exp(-2\alpha^2)]$ with α real. The parameter r can assume the values -1, 0, and 1, which corresponds to an odd coherent state, a coherent state, and an even coherent state, respectively. As we know, the interference terms in (3) rapidly decay to a SM when we include dissipation, so we want to see how different the behaviour of the system would be if the input states are a statistical mixture of the states $|\,\alpha\rangle$ and $|\,-\alpha\rangle$, i.e.,

$$\hat{\rho}^f_{SM}(0) = \frac{1}{2}\Big[|\,\alpha\rangle\langle\alpha\,| + |\,-\alpha\rangle\langle-\alpha\,|\Big] \tag{4}$$

with

$$|\alpha\rangle = \sum_{n=0}^{\infty} q_n |n\rangle = \sum_{n=0}^{\infty} e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \,. \tag{5}$$

The time evolution of the density operator $\hat{\rho}(t)$ at any time $t > 0$ is given by

$$\frac{d\hat{\rho}(t)}{dt} = \frac{-i}{\hbar} [\hat{H}(t), \hat{\rho}(t)] \,. \tag{6}$$

According to Eqs. (3) and (4), the time-dependent analytical solution for the density matrix $\hat{\rho}(t)$ is given by

$$\hat{\rho}(t) = \begin{pmatrix} \hat{\rho}_{ee}(t) & \hat{\rho}_{eg}(t) \\ \hat{\rho}_{ge}(t) & \hat{\rho}_{gg}(t) \end{pmatrix} \,, \tag{7}$$

with

$$\hat{\rho}_{ee}(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_{n,m} A_n(t) A_m^*(t) |n\rangle\langle m| \,,$$

$$\hat{\rho}_{gg}(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_{n,m} B_n(t) B_m^*(t) |n+1\rangle\langle m+1| \,,$$

$$\hat{\rho}_{eg}(t) = (\hat{\rho}_{ge}(t))^+ = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_{n,m} A_n(t) B_m^*(t) |n\rangle\langle m+1| \,,$$

where for the statistical mixture state

$$C_{n,m} = \frac{1}{2} q_n q_m^* [1 + (-1)^{n+m}]$$

and for the superposition states

$$C_{n,m} = \frac{1}{A} q_n q_m^* [(1 + r(-1)^n)(1 + r(-1)^m)] \,,$$

with the coefficients $A_n(t)$ and $B_n(t)$ being given by

$$\begin{aligned} A_n(t) &= \cos(\lambda\theta(t) F_n) \,, \\ B_n(t) &= -i \sin(\lambda\theta(t) F_n) \,. \end{aligned} \tag{8}$$

Here $F_n = f(n+1)\sqrt{n+1}$, and for a particular choice of the atomic motion velocity $\nu = \frac{\lambda L}{\pi}$ and $\theta(t)$ is given by

$$\theta(t) = \int_0^t g(t')dt' = \frac{1}{p\lambda} \sin(p\lambda t) \,. \tag{9}$$

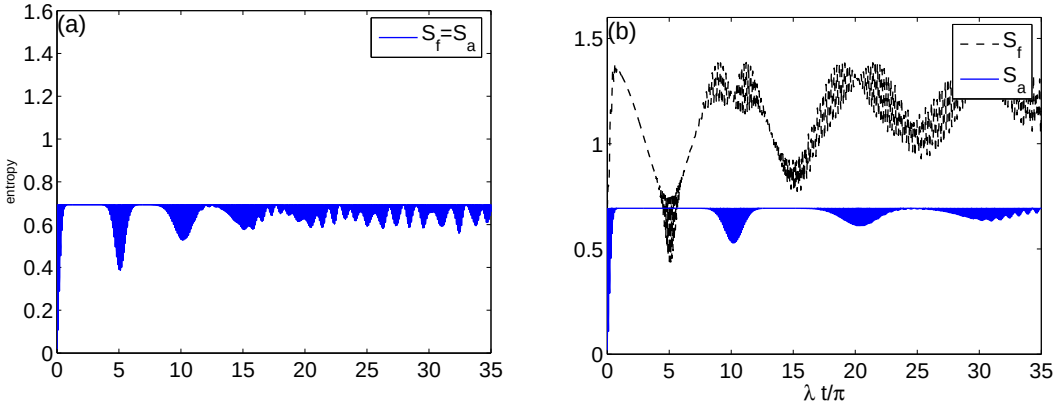

FIG. 1: The entropy for the atom S_a (solid curve) and the field entropy S_f (dashed curve) as a function of the scaled time λt of the particle initially prepared in the excited state and the field initially prepared in: (a) a superposition state ($r = 1$ (even coherent state)) and (b) a statistical mixture (SM) of coherent states $|\alpha\rangle$ and $|-\alpha\rangle$, ($\alpha = 5$), the atom at rest, and $f(n) = \hat{I}$.

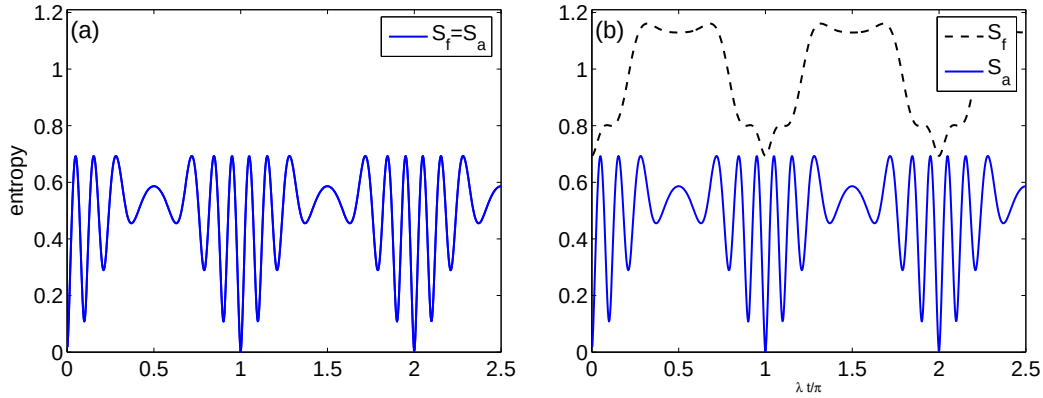

FIG. 2: The same as in Fig. 1 but with the atomic motion parameter $p = 1$.

When we let $p \to 0$ in the above formalism we get the case of the rest atom. The quantum dynamics described by the Hamiltonian (1) leads to an entanglement between the field and the atom. In order to derive a calculation formalism of the partial entropies, we must obtain the eigenvalues of the reduced field density operator and the reduced particle density operator. By using Eq. (7) the reduced field (atom) density operator $\rho_{f(a)}(t) = Tr_{a(f)}\rho(t)$, we investigate the properties of the entropies (S_a, S_f) and entanglement in the framework of the JC model.

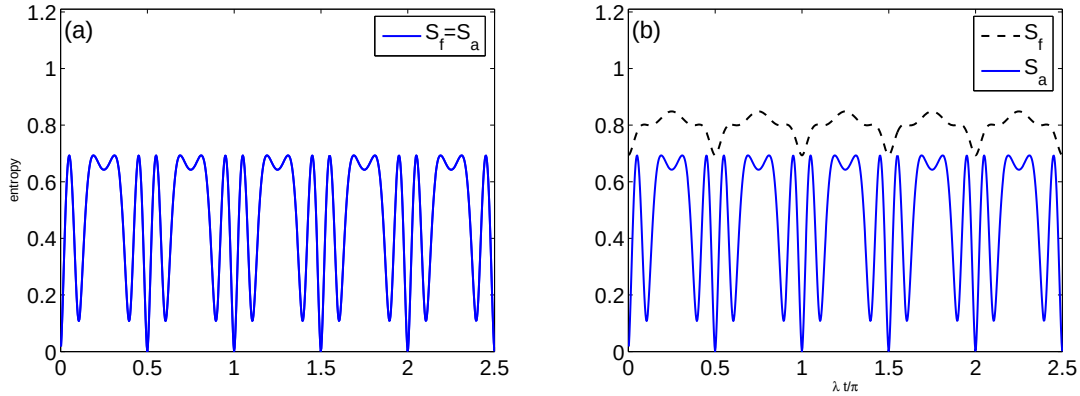

FIG. 3: The same as in Fig. 1 but with the atomic motion parameter $p = 2$.

III. DEGREE OF ENTANGLEMENT

Despite the fact that the basic idea of quantum entanglement was acknowledged almost as soon as quantum theory was discovered, it is only in the last few years that consideration has been given to finding mathematical methods to generally quantify entanglement [4]. In the case of a pure quantum state consisting of two subsystems, a number of widely accepted measures of entanglement are known. However, the question of quantifying the degree of entanglement for general mixed states is still under discussion [21–24]. In this article, we study the evolution of the partial entropies of the atom and field subsystems in the framework of the JC model, considering both the atom at rest and the effects of the atomic motion and field mode structure with different values of the intensity coupling $f(n)$. We will compare the behaviour of the system in the case of having a coherent superposition state SS and a statistical mixture of coherent states SM as the initial field state, when the atomic motion is neglected and when it is considered. Let us now briefly repeat some of the key underlying definitions. The entropy S of a quantum-mechanical system described by the density operator $\hat{\rho}$ is defined as follows:

$$S = -Tr\{\hat{\rho}\ln(\hat{\rho})\}\,, \tag{10}$$

where we have set the Boltzmann constant K equal to unity If $\hat{\rho}$ describes a pure state, then $S = 0$, and if $\hat{\rho}$ describes a mixed state, then $S \neq 0$. The entropies of the atomic and field sub-systems are defined by the corresponding reduced density operators:

$$S_{a(f)} = -Tr_{a(f)}\{\hat{\rho}_{a(f)}\ln\rho_{a(f)}\}\,. \tag{11}$$

By using numerical calculations, we calculated the eigenvalues of the reduced density operator for the particle and the field, and then calculated the entropies to investigate quantum entanglement in the system. Let us now come to specific numerical examples to investigate the influence of a superposition of coherent states of light on the evolution of the entropies

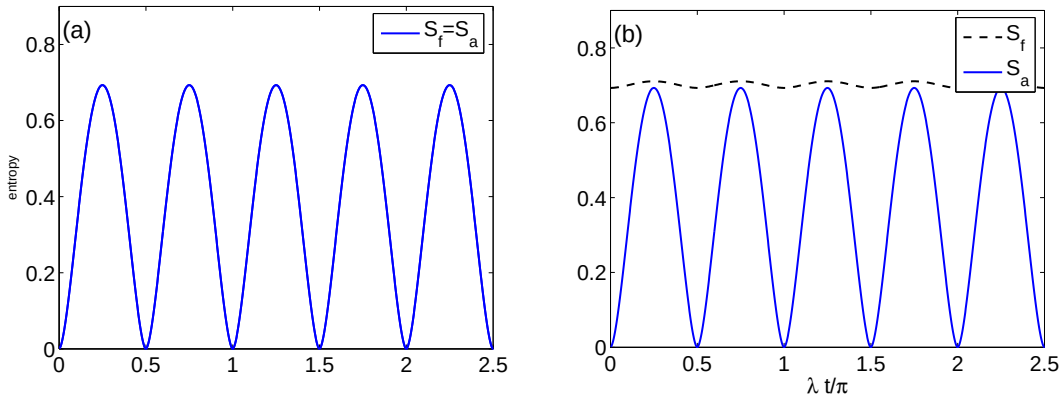

FIG. 4: The entropy for the atom S_a and the field entropy S_f as a function of the scaled time λt of the particle initially prepared in the excited state and the field initially prepared in: (a) even coherent state and (b) a statistical mixture (SM) of coherent states $|\alpha\rangle$ and $|-\alpha\rangle$ ($\alpha = 5$), the atom at rest, and the intensity-dependent atom-field coupling $f(n) = \frac{1}{\sqrt{n}}$.

and entanglement with the effects of atomic motion for a generalized JC-model with a one-quanta transition process. In Figs. (1-9).(a),(b) we see, respectively, the influence of the initial field SS and the SM on the entropy. Our numerical results for the entropy in the SS ($r = 0$ (coherent state) are the same as in Ref. [5–15, 29], where in this case $S_a = S_f$, the maximum entropy is achieved for the evolution of (S_a, S_f) during the first stage of the time evolution, then the entropy evolves to minimum values and the field is completely disentangled from the particle at half of the revival time ($t_R = 2\pi\sqrt{\bar{n}}/\lambda = \frac{10\pi}{\lambda}$). When time increases, the value of the entropy increases. Finally, the entropies (S_a, S_f) remain at their maximum as time increases. Also it can been seen that the behaviors of the entropies are not periodical for the atom at rest and ($f(\hat{n}) = \hat{I}$). While in the superposition state SS ($r = 1$ (even coherent state)) (see Fig. 1(a)), we note that the entropy appears as the SM (the entropy for the atom S_a, see Fig. 1(b) solid curve), and is different from the coherent state case. This effect is due to the interference between the two coherent states in the superposition, and it can be understood by looking at the photon number distribution of the initial fields. Hence this signal gives a clear measure of the remaining degree of coherence between the two components of the cat state, while the signals present in both cases are due to intrinsic revivals of each component individually. In Fig. 1(b), in the statistical mixture states (SM) the entropies (S_a, S_f) are very different from the SS, where in SS the entropies are the same (see Figs. (1-9).(a)), but here in SM the entropy for the field S_f is greater than the entropy for the atom S_a (see Figs. (1-9).(b)), the entropy for the atom remains stable at the maximum value at half of the revival time, while S_f evolves to minimum values.

To visualize the influence of the atomic motion and the field-mode structure on the entropies and entanglement we set different values of p, all the other parameters are the same as in Figs. 1, 4, and 7 for the one photon process. The outcome is presented in

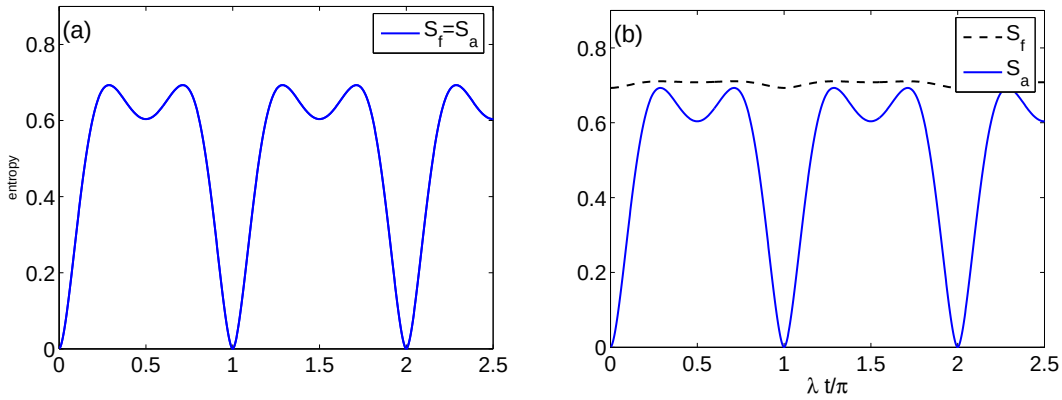

FIG. 5: The same as in Fig. 4 but with the atomic motion parameter $p = 1$.

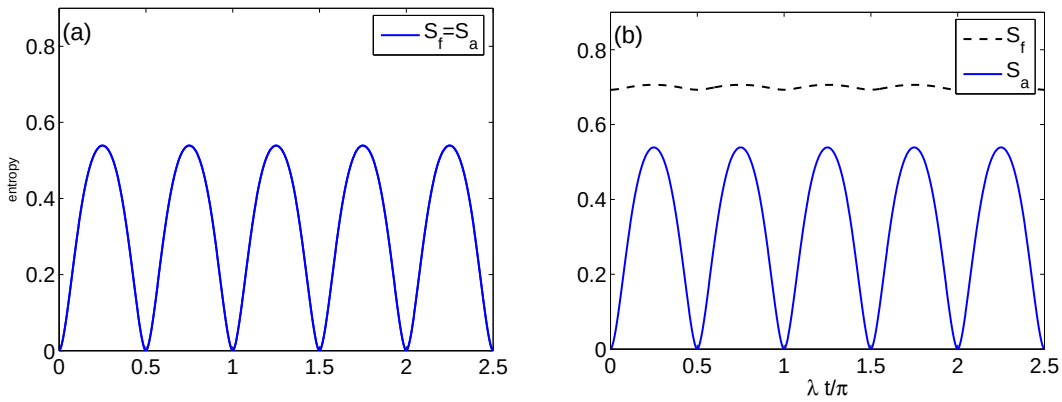

FIG. 6: The same as in Fig. 4 but with the atomic motion parameter $p = 2$.

Figs. 2 and 3 for $f(\hat{n}) = \hat{I}$, and in Figs. 5 and 6 for $f(\hat{n}) = \frac{1}{\sqrt{\hat{n}}}$, and in Figs. 8 and 9 for $f(\hat{n}) = \sqrt{\hat{n}}$. As seen from these figures, we can conclude that the atomic motion leads to the periodic evolution of the entropies and entanglement, and that an increase in the parameter p results in not only a shortening of the evolution period of the field entropy but also a decrease in the amplitude of the field entropy. Physically, all these features can be attributed to the change in the atom-field interaction time due to atomic motion. This is due to the fact that the difference of the field entropy parameters $\hat{\rho}_{ee}(t)$, $\hat{\rho}_{eg}(t)$, $\hat{\rho}_{gg}(t)$ between the two cases in which the atomic motion is neglected and taken into account results from the time functional dependence. The time factor is the scaled time λt when the atomic motion is neglected, and it is $\lambda\theta(t)$ when the atomic motion is taken into account. The evolution behavior of the entropies can be understood simply by considering the relation between $\lambda\theta(t)$, and the scaled time λt. When the atomic motion is not taken into account the evolution of the entropies and entanglement are not periodic (see Fig. 1),

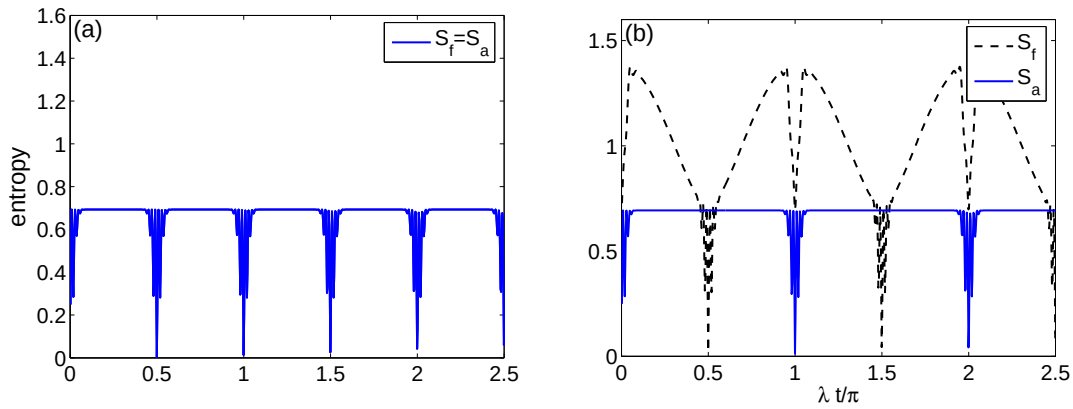

FIG. 7: The entropy for the atom S_a and the field entropy S_f as a function of the scaled time λt of the particle initially prepared in the excited state and the field initially prepared in: (a) a superposition state ($r = 1$ (even coherent state)) and (b) a statistical mixture (SM) of coherent states $|\alpha\rangle$ and $|-\alpha\rangle$ ($\alpha = 5$), the atom at rest, and the intensity-dependent atom-field coupling $f(n) = \sqrt{n}$.

while when the atom is in motion, from Eq. (9), we have $\lambda\theta(t) = \frac{1}{p}\sin(p\lambda t)$. It is observed that $\lambda\theta(t)$ is a periodical function on the scaled time λt with period $2\pi/p$. This periodicity of $\lambda\theta(t)$ of the scaled time λt just leads to the periodicity of evolution of the entropies and entanglement. When $p = 1$, the entropies and entanglement evolve in a period π (see Fig. 2), while when $p = 2$, they evolve in a period $\pi/2$ (see Fig. 3). It is worth noting that in the normal JC model, where the atomic motion is not taken into account, the field entropy and entanglement are as discussed by [5–15], while in the present model we observe that the evolution of the entropies and entanglement depends on the atomic velocity ν, the length of the cavity L, and the field-mode structure parameters. To visualize the influence of the nonlinearities of the intensity-dependent function $f(\hat{n})$ on the entropies and entanglement, we take different forms of the intensity dependent function while the other parameters have the same values. As seen from these figures, we can conclude that the intensity-dependent atom-field coupling $f(\hat{n})$ leads to the periodic evolution of the entropies and entanglement, especially with the intensity atom field function $f(n) = \frac{1}{\sqrt{n}}$ the entropies and entanglement decrease (see Figs. 4-6). In this case the degree of entanglement between the field and the atom reduces, and an almost pure state is reached for the SS state. It is evident that the field and the atom are in pure states when the intensity coupling is taken into consideration and the initial state is SS. While for the SM the state of the atom and the field return to their initial states periodically. It is worth noting that the results presented here are almost similar to those obtained in [12–15], where the von Neumann entropy has been used to quantify the entanglement. For intensity atom-field coupling in the form $f(n) = \sqrt{n}$ but with the atomic motion neglected, see the work [29].

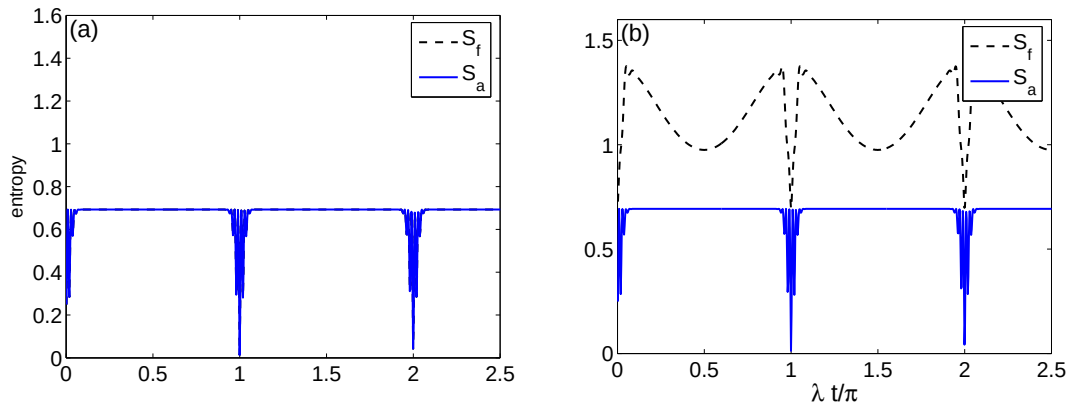

FIG. 8: The same as in Fig. 7 but with the atomic motion parameter $p = 1$.

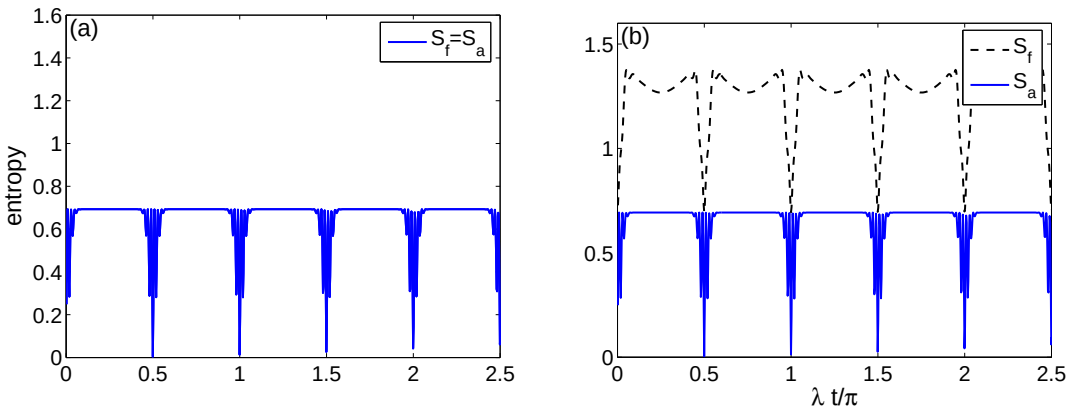

FIG. 9: The same as in Fig. 7 but with the atomic motion parameter $p = 2$.

IV. CONCLUSIONS

The effects of the atomic motion and the field-mode structure, taking into account a different functional dependence for the form of the nonlinear intensity-dependent coupling on the evolution of the entropies and entanglement in the mixed state, have been studied for the two-level atom in the framework of the JC model. By using a numerical method we calculated the partial entropies of the atom and field subsystem. We compared the behaviour of the system with a coherent superposition state and a statistical mixture of coherent states as an initial field state. In the SS ($r = 1$ (even coherent state)) the entropies evolved at the same time to the minimum values and the field and the particle were strongly disentangled. Also in the statistical mixture state (SM) we show the entropies ($S_a \neq S_f$); the entropy for the atom is sustained at the maximum values at half of the revival time, while S_f evolved to minimum values. We can conclude that the initial state plays an important

role in the evolution of the entropies and entanglement. However, atomic motion leads to the periodic evolution of the entropies and entanglement. Increasing the parameter p results not only in shortening the evolution period of the field entropy but also in decreasing the amplitude of the field entropy. With the intensity atom-field coupling in the form $f(n) = \frac{1}{\sqrt{n}}$ the entropies and entanglement decrease.

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