On t/x-Dependent Conservation Laws of the Generalized nth-Order KdV Equation

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By means of a direct algebraic method, new $(n-1)$th-order conservation laws that depend not only on u and its derivatives but also explicitly on t/x are constructed for the nth-order KdV equations.

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I. INTRODUCTION

As is well known, the classical nth-order ($n = 3, 5, 7, 9$) KdV equations have infinitely many conservation laws; most of them are polynomials that only depend on $u\,(u = u(x,t))$ and its derivatives, and do not depend explicitly on t/x. The classical KdV equations, as a matter of fact, do have such conservation laws that depend explicitly on t/x. The aim of this paper is to construct the $(n-1)$th-order (highest derivative) t/x-dependent conservation laws for the nth-order KdV equations. To reach this end, we combine the direct algebraic method described below and the method presented in [1] with the assistance of the computer algebraic system *Maple*.

II. METHOD OF FINDING THE t/x-DEPENDENT CONSERVATION LAWS

The nth-order KdV equations with the general form

$$u_t = u_{nx} + H(u, u_x, \cdots, u_{(n-1)x}) \equiv F(u)\,, \tag{1}$$

where $u_{ix} = \partial^i u/\partial x^i$, and H is a specific polynomial in u and its derivatives, is invariant under the following scaling transformation

$$(t,\, x,\, u) \mapsto (\mu^{-n}t,\, \mu^{-1}x,\, \mu^2 u)\,. \tag{2}$$

Hence the weight [3] value of the dependent variable u, denoted by $\omega(u)$, is 2, which is useful for us.

Combining the method given in [1], the following proposition, proved in [1], and its corollary, allow one to obtain the t/x-dependent conservation laws. As usual, D_x and D_t are total derivatives with respect to x and t, respectively.

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Proposition If $D_x^{-1}F$, denoted by ρ, is a conserved density of the evolution equation $u_t = F$, then $tD_x^{-1}F + xu$ is also one of its conserved densities.

Corollary If J is a conserved flux corresponding to the conserved density $D_x^{-1}F$, then $tJ - xD_x^{-1}F$ is the associated conserved flux of $tD_x^{-1}F + xu$.

Then the basic algorithm to construct the t/x-dependent conservation laws (see [1] for its definition) for the nth-order KdV equations is fairly straightforward:

- *Step 1: Determine the general form of ρ.*

  Find the building blocks in u and its x-derivatives of ρ that satisfy the equation $\rho = D_x^{-1}F$. This is a crucial step. In detail, the procedure proceeds as follows:

  1. Determine the rank [1, 3], R, of ρ. Assuming that $\rho = D_x^{-1}F$ is a conserved density of Eq. (1), then we have

     $$F = D_x\rho\,. \tag{3}$$

     From Eq. (3) along with Eq. (1), we know that the highest derivative of ρ is $n-1$ and ρ should contain the term of $u_{(n-1)x}$. Since $\omega(\partial/\partial x) = 1$ [1, 3], R is then obtained as

     $$R = (n-1)\omega(\partial/\partial x) + \omega(u) = n+1\,.$$

  2. Form the basis set B. It consists of such elements as $[M, \mathrm{Rank}(M)]$, where M is one of the monomials with rank R or less, by taking all appropriate combinations of different powers of u; $\mathrm{Rank}(M)$ is the rank of the monomials M.
  3. Form the set Q that contains all monomials in u and its x-derivatives with rank R. To do this, for each element in the set B, compute $l_i = R - \mathrm{Rank}(M_i)$, which forms a list L. Then compute the x-derivative of M_i up to l_i, such that the new generating monomials exactly have rank R. Gather the new generating monomials in the set Q.
  4. Linear combination of the monomials in the set Q with constant coefficients c_i's directly yield the general form of ρ. In contrast to the method of constructing the polynomial type conservation laws in [1], no terms are removed here.

  Carrying on with the 3rd-order KdV equation

  $$u_t = p\,uu_x + u_{3x} \equiv F\,, \tag{4}$$

  where p is an arbitrary constant, apparently the ρ that satisfies Eq. (3) should contain the term of u_{2x}. Thus the rank of ρ is 4, and the basis set $B = \{[u, 2], [u^2, 4]\}$. Easily, we get the list $L = [2, 0]$. Then we compute the various x-derivatives of u and u^2 up

to 2 and 0, respectively, such that the new generating monomials exactly have rank 4. Therefore, we have

$$\frac{d^2}{dx^2}u = u_{2x}, \quad \frac{d^0}{dx^0}(u^2) = u^2 .$$

Gathering the monomials on the right hand side of the above two expressions, we derive the set $Q = \{u_{2x}, u^2\}$. Hence, the general form of ρ is

$$\rho = c_1 u^2 + c_2 u_{2x} .$$

- *Step 2: Determine the unknown coefficients in ρ.*

  Compute $D_x\rho$ and, in view of Eq. (3), determine the c_i's. For example, for Eq. (4), we have $p\,uu_x + u_{3x} = 2c_1 uu_x + c_2 u_{3x}$. Thus, $\{c_1 = p/2,\ c_2 = 1\}$, and

$$\rho = \frac{1}{2}p\,u^2 + u_{2x} . \tag{5}$$

- *Step 3: Rewrite ρ as its equivalent form $\tilde{\rho}$.*

  First, rewrite ρ into the form

$$\rho = \triangle + [\bullet]' , \tag{6}$$

  as far as one can, where $\triangle$ is a polynomial in u and its derivatives and the symbol $'$ denotes the derivative of the expression $\bullet$ with respect to x. For example, the monomial $uu_{3x}u_{5x} = (uu_{3x}u_{4x})' - (uu_{4x}^2 + u_x u_{3x} u_{4x})$ and $u_x u_{3x} u_{4x} = (u_x u_{3x}^2)' - u_{2x}u_{3x}^2$. Thus the monomial $uu_{3x}u_{5x}$ is finally reexpressed as $(uu_{3x}u_{4x} - u_x u_{3x}^2)' + u_{2x}u_{3x}^2 - uu_{4x}^2$. For a polynomial ρ in u and its derivatives, do the same thing repeatedly. ρ can be rewritten as (6). Then, from the knowledge that the conserved densities are equivalent if they only differ by a total x-derivative we know that ρ is equivalent to

$$\tilde{\rho} = \triangle .$$

  Taking Eq. (4) as an example, Eq. (5) is equivalent to $\tilde{\rho} = \frac{1}{2}p\,u^2$ or u^2.

- *Step 4: Verify whether $\tilde{\rho}$ is a conserved density or not.*

  As a matter of fact, u^2 is indeed a conserved density for Eq. (4), because Eq. (4) can be written as

$$D_t(\tilde{\rho}) + D_x(\tilde{J}) = D_t(u^2) + D_x(-\frac{2}{3}p\,u^3 - 2uu_{2x} + u_x^2) = 0 . \tag{7}$$

  By now, we verify that Eq. (5) is a conserved density of Eq. (4). For other higher order KdV equations, we can verify whether or not the equivalent form $\tilde{\rho}$ is a conserved density by means of the procedure `CONSLAW` [1].

- *Step 5: Determine the form of the t/x-dependent conserved density $\bar{\rho}$.*

  We point out that, although in the sense of a conservation law ρ is equivalent to $\tilde{\rho}$, the t/x-dependent conserved density, denoted by $\bar{\rho}$, can not be obtained directly by the above proposition, i.e. $\bar{\rho} \neq t\tilde{\rho} + xu$, instead, it should be $\bar{\rho} = t\rho + xu$. Hence, we conclude that the previously known expression, $\frac{1}{2}p\,t\,u^2 + xu$, which is regarded as a conserved density of Eq. (4) by the authors of [2, 3], is not one. The radical reason lies in $D_x^{-1}\tilde{\rho} \neq F$.

- *Step 6: Determine the corresponding conserved flux of $\bar{\rho}$.*

  For Eq. (4), by using the procedure `CONSLAW` we find the conserved flux corresponding to $\rho = \frac{1}{2}p\,u^2 + u_{2x}$ to be

  $$J = -\frac{1}{2}p\,u_x^2 - \frac{1}{3}p^2\,u^3 - 2p\,uu_{2x} - u_{4x}\,.$$

  Thus, by the above proposition and the corollary, we obtain the only t/x-dependent conservation law for Eq. (4) as

  $$D_t[t(\frac{1}{2}p\,u^2+u_{2x})+xu]+D_x[t(-\frac{1}{2}p\,u_x^2-\frac{1}{3}p^2\,u^3-2p\,uu_{2x}-u_{4x})-x(\frac{1}{2}p\,u^2+u_{2x})] = 0\,,$$

  which is new, and is reported here for the first time.

III. t/x-DEPENDENT CONSERVATION LAW FOR THE GENERALIZED 5th-ORDER KdV EQUATION

The generalized 5th-order KdV equation

$$u_t = au^2u_x + bu_xu_{2x} + cuu_{3x} + u_{5x} \equiv F(u)\,, \tag{8}$$

with constant parameters a, b, c, and $u_{kx} = \partial^k/\partial x^k$, includes four well known special cases: for $(a, b, c) = (10, 20, 30)$, $(20, 40, 120)$, $(30, 60, 270)$ it is the Lax equation [4]; for $(a, b, c) = (-15, -15, 45)$, $(5, 5, 5)$, $(30, 30, 30)$ it is an equation, called the SK equation, due to Sawata and Kotera [5], and also the case $(5, 5, 5)$ due to Dodd and Gibbon [6]; for $(a, b, c) = (10, 25, 20)$, $(30, 75, 180)$ it is an equation, called KK due to Kaup [7] and Kupershmidt; for $(a, b, c) = (3, 6, 2)$, it is an equation, called Ito, due to Ito [8].

By a simple transformation $u = u/a$, Eq. (8) is reduced to

$$u_t + uu_{3x} + \alpha u_xu_{2x} + \beta u^2u_x + u_{5x} = 0\,, \tag{9}$$

where $\alpha = b/a$ and $\beta = c/a^2$. Especially, the Lax equation is obtained from Eq. (9) for $\alpha = 2, \beta = \frac{3}{10}$. The Sawada, KK, and Ito equations correspond to $(\alpha, \beta) = (1, \frac{1}{5}), (\frac{5}{2}, \frac{1}{5}), (2, \frac{2}{9})$ respectively.

Through the same analysis as used above we find the general form of ρ that satisfies Eq. (3) to be

$$\rho = c_1 u^3 + c_2 u_x^2 + c_3 u_{4x} + c_4 u u_{2x}\,.$$

In view of $D_x^{-1}F = \rho$, i.e. $F = D_x\rho$, we obtain

$$\rho = \frac{\beta}{3}u^3 + \frac{\alpha-1}{2}u_x^2 + u_{4x} + u u_{2x}\,,$$

which can also be written as

$$\rho = \frac{\beta}{3}u^3 + \frac{\alpha-3}{2}u_x^2 + (u_{3x} + u u_x)'\,,$$

and is equivalent to

$$\tilde{\rho} = \frac{\beta}{3}u^3 + \frac{\alpha-3}{2}u_x^2\,, \quad \text{or} \quad \tilde{\rho} = \frac{2\beta}{3(\alpha-3)}u^3 + u_x^2\,.$$

By using `CONSLAW` [1] we verify that $\tilde{\rho}$ is indeed a conserved density for Eq. (9) under the parameter constraints

$$\{\beta = \frac{(2\alpha-1)(\alpha-3)}{10}, \quad \alpha : \text{free}\}\,.$$

Hence, ρ is a conserved density that satisfies the equation $D_x^{-1}F = \rho$. Thus we get the t/x-dependent conserved density as

$$\bar{\rho} = t[\frac{(2\alpha-1)(\alpha-3)}{30}u^3 + \frac{\alpha-1}{2}u_x^2 + u u_{2x} + u_{4x}] + xu\,.$$

For brevity, the corresponding conserved flux is omitted here.

IV. t/x-DEPENDENT CONSERVATION LAW FOR THE GENERALIZED 7th-ORDER KdV EQUATION

The generalized 7th-order KdV equation,

$$u_t = F = a u^3 u_x + b u_x^3 + c u u_x u_{2x} + d u^2 u_{3x} + e u_{2x} u_{3x} + f u_x u_{4x} + u u_{5x} + u_{7x}\,, \tag{10}$$

with six constant parameters a, b, c, d, e, f, and $u_{kx} = \partial^k/\partial x^k$ includes three cases that are of particular interest: (i) $(a,b,c,d,e,f) = (\frac{5}{98}, \frac{5}{14}, \frac{10}{7}, \frac{5}{14}, 5, 3)$ (Lax equation) [8]; (ii) $(a,b,c,d,e,f) = (\frac{4}{147}, \frac{1}{7}, \frac{6}{7}, \frac{7}{2}, 3, 2)$ (SK-Ito equation) [8]; (iii) $(a,b,c,d,e,f) = (\frac{4}{147}, \frac{5}{14}, \frac{9}{7}, \frac{2}{7}, 6, \frac{7}{2})$ (found in [1]).

By analysis, we find that the rank of ρ that satisfies $D_x^{-1}F = \rho$ is 8. The general form of ρ is

$$\rho = c_1 u u_x^2 + c_2 u_{2x}^2 + c_3 u_x u_{3x} + c_4 u^2 u_{2x} + c_5 u u_{4x} + c_6 u^4 + c_7 u_{6x}\,. \tag{11}$$

Then, in view of the equation $F = D_x\rho$, we get

$$\rho = buu_x^2 + \frac{e-f+1}{2}u_{2x}^2 + (f-1)u_x u_{3x} + \frac{c-2b}{2}u^2 u_{2x} + uu_{4x} + \frac{a}{4}u^4 + u_{6x}\,, \tag{12}$$

which can also be written as

$$\rho = (3b-c)uu_x^2 + \frac{e-3f+5}{2}u_{2x}^2 + \frac{a}{4}u^4 + [(f-2)uu_{2x} + \frac{c-2b}{2}u^2 u_x + u_{5x}]'\,, \tag{13}$$

and is equivalent to

$$\tilde{\rho} = (3b-c)uu_x^2 + \frac{e-3f+5}{2}u_{2x}^2 + \frac{a}{4}u^4\,. \tag{14}$$

By using `CONSLAW` we verify that the $\tilde{\rho}$ that satisfies the equation $D_x^{-1}F = \rho$ is a conserved density of Eq. (10) under the parameter constraints

$$\{a = \tfrac{(4-f)(10f^2-45f+49c+20)}{882},\ b = \tfrac{-9\,f+2\,f^2+4+14\,c}{42},\ d = \tfrac{7\,c-4+9\,f-2\,f^2}{42},\ e = 2f-1,\ c,\ f : \text{free}\}.$$

Thus, by the above proposition, the t/x-dependent conservation law is

$$\begin{aligned}\hat{\rho} = t\{&\tfrac{(2f-1)(f-4)}{14}uu_x^2 + \tfrac{4-f}{2}u_{2x}^2 + \tfrac{(4-f)(10f^2-45f+49c+20)}{3528}u^4 + [(f-2)uu_{2x}\\ &+ \tfrac{7c+9f-2f^2-4}{42}u^2u_x + u_{5x}]'\} + xu\,.\end{aligned}$$

The corresponding conserved flux is also omitted here.

V. t/x-DEPENDENT CONSERVATION LAW FOR THE 9th-ORDER KdV EQUATION

For the 9th-order KdV equation, we consider its usual form

$$\begin{aligned}u_t = {}& 35uu_{2x}u_{3x} + \tfrac{217}{6}u_x u_{2x}^2 + 21uu_x u_{4x} + \tfrac{35}{6}uu_x^3 + \tfrac{35}{3}u^2u_x u_{2x} + 42u_{3x}u_{4x} + 28u_{2x}u_{5x}\\ &+12u_x u_{6x} + 3uu_{7x} + \tfrac{161}{6}u_x^2u_{3x} + \tfrac{7}{2}u^2u_{5x} + \tfrac{35}{18}u^3u_{3x} + \tfrac{35}{72}u^4u_x + u_{9x} \equiv F(u)\,.\end{aligned} \tag{15}$$

To obtain its t/x-dependent conservation law, we first compute the polynomial type conserved density of order 8, denoted by ρ, which depends only on u and its derivatives. Since Eq. (15) has the same scaling property as Eq. (1), the rank of ρ is 10. Hence, according to the above algorithm, we find the general form of ρ that satisfies the equation $D_x^{-1}F = \rho$ to be

$$\begin{aligned}\rho = {}& c_1u^2u_x^2 + c_2u_{3x}^2 + c_3u_{2x}u_{4x} + c_4u^5 + c_5u_xu_{5x} + c_6uu_{6x} + c_7u_x^2u_{2x} + c_8uu_{2x}^2 + c_9uu_xu_{3x}\\ &+ c_{10}u^2u_{4x} + c_{11}u^3u_{2x} + c_{12}u_{8x}\,.\end{aligned}$$

In view of the equation $D_xF = \rho$, we can determine the unknown c_i's and then get

$$\begin{aligned}\rho = {} & \frac{35}{12}u^2u_x^2 + \frac{23}{2}u_{3x}^2 + 19u_{2x}u_{4x} + \frac{7}{72}u^5 + 9u_xu_{5x} + 3uu_{6x} + \frac{77}{6}u_x^2u_{2x} + \frac{21}{2}uu_{2x}^2 \\ & + 14uu_xu_{3x} + \frac{7}{2}u^2u_{4x} + \frac{35}{18}u^3u_{2x} + u_{8x}\,,\end{aligned} \tag{16}$$

which can be written as

$$\begin{aligned}\rho = {} & \frac{7}{72}u^5 - \frac{35}{12}u^2u_x^2 - \frac{3}{2}u_{3x}^2 + \frac{7}{2}uu_{2x}^2 + [6u_xu_{4x} + 13u_{2x}u_{3x} + 3uu_{5x} + \frac{35}{18}u_x^3 + 7uu_xu_{2x} \\ & + \frac{7}{2}u^2u_{3x} + \frac{35}{18}u^3u_x + u_{7x}]'\,,\end{aligned} \tag{17}$$

and is equivalent to

$$\tilde{\rho} = \frac{7}{72}u^5 - \frac{35}{12}u^2u_x^2 - \frac{3}{2}u_{3x}^2 + \frac{7}{2}uu_{2x}^2 \quad \text{or} \quad \tilde{\rho} = u^5 - 30u^2u_x^2 - \frac{108}{7}u_{3x}^2 + 36uu_{2x}^2\,.$$

As before, using the procedure `CONSLAW`, we can easily verify that $\tilde{\rho}$ is indeed a conserved density for Eq. (15). This shows that (16), which satisfies the equation $D_xF = \rho$, is also a conserved density of Eq. (15). Hence, the 8th-order t/x-dependent conserved density $\bar{\rho}$ for Eq. (15) is generated from Eq. (16) or (17) along with the expression $t\rho + xu$.

VI. CONCLUSION

In summary, combing the approach described in [1] and the direct algebraic method presented above, we obtain the $(n-1)$th-order t/x-dependent conservation laws for the generalized nth-order KdV equations. As a matter of fact the method is applicable for any nonlinear evolution equation $u_t = F(u, u_x, u_{2x}, \cdots,)$, as long as we can find a conserved density ρ that satisfies the equation $D_x^{-1}F = \rho$.

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