

A Micro-Stochastic Equation for Quantum Mechanics

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A stochastic equation named as Micro-Stochastic equation is proposed to describe the motion of a microscopic object assuming that the object is interacting with a background field through a random force and a associated friction force, undergoing a Markoff process. It is shown that in one limit, this dynamical equation leads to the Schrödinger equation in quantum mechanics when the friction coefficient $\beta \rightarrow \infty$; in other limit, it reduces to classical mechanics when $\hbar/m \rightarrow 0$. New physics might be expected when \hbar/m is significant and β is not too large. In this stochastic approach, the physical quantities of a microscopic object can take any possible values like a classical object; no quantization is required. The nature of eigen values and quantization of physical quantities in quantum mechanics are discussed.

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I. Introduction

Since quantum mechanics conceptually is so different from the classical mechanics, there have been many efforts trying to understand the quantum phenomena within the framework of classical mechanics. One of the approaches is attempting to explain the motion of a microscopic object as a stochastic process [1, 2]. The hypothesis is to assume that a microscopic system is in fact an open system, constantly interacting with environment (the vacuum or a unknown background field), just like Brownian particles hit by the surrounding molecules randomly, thus causing fluctuation, and can not be described by any deterministic theory.

In the well known hidden variable theory [3], Bohm has shown that solving Schrödinger equation in quantum mechanics is equivalent to solve the following two equations:

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + (U + V_Q) = 0 \quad (1)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\vec{v}\rho) = 0 \quad (2)$$

with

$$\psi = R(\mathbf{x}, t) e^{\frac{i}{\hbar} S(\mathbf{x}, t)}, \quad \rho = \psi^* \psi = R(\mathbf{x}, t)^2 \quad (3)$$

$$\vec{v} = \nabla S(\mathbf{x}, t)/m, \quad V_Q = \frac{\hbar^2}{2m} \frac{\nabla^2 R}{R} \quad (4)$$

Note that Eq. 1 may be considered as the Hamilton-Jacobi equation in classical mechanics with $S(\mathbf{x}, t)$ as the Hamilton principle function and $V = U + V_Q$ as an effective potential. Thus the motion of a microscopic object might be understood within the framework of classical mechanics. There are only two differences: one is the necessity of adding a so called quantum potential V_Q , the other is the fluctuation of the initial condition. Both may be interpreted as the consequence of the existence of the uncontrollable random interaction with the environment. However, Bohm's theory did not give precisely what is the random force and how it leads to the quantum potential. Meanwhile, the trajectories obtained from Eq. 1 can not be the true paths that a particle really pass through. Because, the trajectory predicted by Eq. 1 is completely deterministic for every given initial condition, which is in contradiction to the nature of a stochastic process with a random force acting on the object.

In 1966, Nelson [4] was first time succeed in deriving Schrödinger equation by really treating the motion of a microscopic object as a Markoff stochastic process. He found that the random force is adequately described by the Wiener process $W(t)$ as that has been used in Brownian motion and other classical stochastic processes. The $dW(t)$ is Gaussian, mutually independent, and has following properties:

$$E_t[dW_i] = 0, \quad E_t[dW_i(t) dW_j(t)] = 2D\delta_{ij}dt, \quad (5)$$

where E_t denotes the conditional expectation (average) at time t , dW_i the i -th component of the random force $dW(t)$, and D the diffusion coefficient in the vacuum. In order to obtain the Schrödinger equation, it was found that

$$D = \frac{\hbar}{2m} \quad (6)$$

Thus, Nelson's work seems to confirm the hypothesis that quantum mechanics is equivalent to a classical stochastic process. In Nelson's paper he actually use the title "Derivation of Schrödinger equation from Newtonian Mechanics". However there are still debates on the subject whether indeed quantum mechanics can be formulated as a classical Markoff process or not [5].

In this paper we shall indicate that although Nelson's work is a significant step toward to the deeper understanding of quantum mechanics, it is not yet completed. To complete the Nelson's approach, a stochastic dynamical equation is proposed. We confirm that quantum mechanics is indeed equivalent to a Markoff stochastic process, but not the classical one. The dynamics of a microscopic object is fundamentally different from the classical Brownian motion. In the following section we shall begin with a brief review of Nelson's work, and indicate what was incomplete in his approach. Then in Section III, a microscopic stochastic dynamical equation will be proposed. We shall show that in one limit, it leads to the Schrödinger equation in quantum mechanics when the friction coefficient $\beta \rightarrow \infty$; while in other limit, it reduces to classical mechanics when $\hbar/m \rightarrow 0$. In Section IV, numerical simulations of a two dimensional rotor is presented to demonstrate

the equivalence between quantum mechanics and the microscopic stochastic approach in $\beta \rightarrow \infty$ limit. The nature of eigen values and quantization of physical quantities is also discussed. Finally a brief summary is given in Section V.

II. The Nelson's approach

In the Nelson's approach, It was started from the stochastic dynamical equation of the Brownian motion (see page 71 in[6]),

$$\begin{aligned} d\mathbf{X}(t) &= \mathbf{v}(t)dt \\ d\mathbf{v}(t) &= -\beta\mathbf{v}(t)dt + \beta\mathbf{b}(\mathbf{x}(t),t)dt + \beta d\mathbf{W}(t) \end{aligned} \quad (7)$$

where $d\mathbf{W}(t)$ is a Wiener process with diffusion coefficient D (see Eq. (5)), and

$$\mathbf{b}(\mathbf{x}(t),t) = \frac{1}{\beta}\mathbf{K}(\mathbf{x}(t)), \quad \mathbf{K}(\mathbf{x}(t)) = -(1/m)\nabla U(\mathbf{x}(t)) \quad (8)$$

Nelson was able to prove that the solution, $\mathbf{x}(t)$ of the coupled stochastic dynamical equations Eq. (7), is the same as the solution of the following equation, when the friction coefficient $\beta \rightarrow \infty$:

$$d\mathbf{x}(t) = \mathbf{b}(\mathbf{x}(t),t)dt + d\mathbf{W}(t) \quad (9)$$

Next, it was realized that it is necessary to distinguish forward and backward velocities. The mean forward and backward velocities are defined as follows:

$$D_t^{(+)}\mathbf{x}(t) = \lim_{\Delta t \rightarrow 0+} E_t \frac{\mathbf{x}(t+\Delta t) - \mathbf{x}(t)}{\Delta t} = \lim_{\Delta t \rightarrow 0+} \frac{\langle \mathbf{x}(t+\Delta t) \rangle - \mathbf{x}(t)}{\Delta t} \quad (10)$$

$$D_t^{(-)}\mathbf{x}(t) = \lim_{\Delta t \rightarrow 0+} E_t \frac{\mathbf{x}(t) - \mathbf{x}(t-\Delta t)}{\Delta t} = \lim_{\Delta t \rightarrow 0+} \frac{\mathbf{x}(t) - \langle \mathbf{x}(t-\Delta t) \rangle}{\Delta t} \quad (11)$$

where E_t denotes the conditional expectation given by the state at time t . The symbol $\langle \dots \rangle$ means doing average with respect to the paths under the condition that they all go through $\mathbf{x}(t)$ at time t . The symbol $0+$ means that Δt tends to zero through positive values. Note that a conditional expectation is not a number but a function of \mathbf{x} and t . If $\mathbf{x}(t)$ is differentiable as in the case with no random force, then of course, $D_t^{(+)}\mathbf{x}(t) = D_t^{(-)}\mathbf{x}(t) = d\mathbf{x}/dt$; but since $\mathbf{x}(t)$ is a random variable, $D_t^{(\pm)}\mathbf{x}(t)$ may not be the same. Therefore, Eq. (9) should be separated into two, forward and backward, equations:

$$d\mathbf{x}(t) = \mathbf{b}(\mathbf{x}(t),t)dt + d\mathbf{W}_+(t) \quad (12)$$

$$d\mathbf{x}(t) = \mathbf{b}_*(\mathbf{x}(t),t)dt + d\mathbf{W}_-(t) \quad (13)$$

where $d\mathbf{W}_+(t)$ ($d\mathbf{W}_-(t)$) are independent of all $\mathbf{x}(s)$ with $s \leq t$ ($s \geq t$). It is clear that Eq. (13) is the time reversal of Eq. (12); $\mathbf{b}(\mathbf{x}(t),t)$ and $\mathbf{b}_*(\mathbf{x}(t),t)$ are the mean forward and backward velocities, respectively, *i. e.*

$$D_t^{(+)}\mathbf{x}(t) = \mathbf{b}(\mathbf{x}(t), t), \quad D_t^{(-)}\mathbf{x}(t) = \mathbf{b}_*(\mathbf{x}(t), t) \quad (14)$$

In addition to this, the mean second derivative of a stochastic process is defined to be

$$\mathbf{a}(\mathbf{t}) = \frac{1}{2} \left[D_t^{(+)} D_t^{(-)} \mathbf{x}(t) + D_t^{(-)} D_t^{(+)} \mathbf{x}(t) \right] \quad (15)$$

From the dynamical equation of Brownian motion (Eq. (7) and under the condition of Eq. (8)), it can be shown that $\mathbf{a}(\mathbf{t}) = \mathbf{K}(\mathbf{x}(t))$. Thus,

$$\frac{1}{2} \left[D_t^{(+)} \mathbf{b}_*(\mathbf{x}(t), t) + D_t^{(-)} \mathbf{b}(\mathbf{x}(t), t) \right] = \mathbf{K}(\mathbf{x}(t)) \quad (16)$$

Let $\rho(\mathbf{x}, t)$ be the probability density of finding particles at position \mathbf{x} and time t , Eq. (12) and Eq. (13) lead to the following forward and backward Fokker-Planck equations:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\mathbf{b}\rho) + D\nabla^2 \rho \quad (17)$$

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\mathbf{b}_*\rho) - D\nabla^2 \rho \quad (18)$$

The average of Eq. (17) and Eq. (18) yields the equation of continuity

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\bar{\mathbf{v}}\rho) = 0, \quad \bar{\mathbf{v}} = \frac{1}{2}(\mathbf{b} + \mathbf{b}_*) \quad (19)$$

where $\bar{\mathbf{v}}$ is the current velocity. Subtracting Eq. (18) from Eq. (17) one yields the following relations:

$$\bar{\mathbf{u}} = D\nabla(\ln \rho), \quad \text{ii} = \frac{1}{2}(\mathbf{b} - \mathbf{b}_*) \quad (20)$$

since $\bar{\mathbf{u}}$ depends on the gradient of probability density ρ , it is called the osmotic velocity.

Note that for a function of \mathbf{x} and t , $f(\mathbf{x}(t), t)$, to calculate $D_t^{(+)}f(\mathbf{x}(t), t)$ and $D_t^{(-)}f(\mathbf{x}(t), t)$ is different from the ordinary calculus. Expanding $f(\mathbf{x}(t), t)$ in a Taylor series,

$$\begin{aligned} df(\mathbf{x}(t), t) &= \frac{\partial}{\partial t} f(\mathbf{x}(t), t) dt + d\mathbf{x}(t) \cdot \nabla f(\mathbf{x}(t), t) \\ &+ \frac{1}{2} \sum_{i,j} dx_i dx_j \frac{\partial^2}{\partial x_i \partial x_j} f(\mathbf{x}(t), t) + \dots \end{aligned} \quad (21)$$

When the conditional average is taken, by Eq. (12) and Eq. (13), one may replace $d\mathbf{x}(t)$ by $\mathbf{b}dt$ (\mathbf{b}_*dt) since $E_t[d\mathbf{W}] = \mathbf{0}$. However, the second order term in the expansion has to be taken into account, because $E_t[dW_i dW_j]$ has a mean value which is proportional to \mathbf{dt} (see Eq. (5)). Therefore,

$$D_t^{(+)}f(\mathbf{x}(t), t) = \left(\frac{\partial}{\partial t} + \mathbf{b} \cdot \nabla + D\nabla^2 \right) f(\mathbf{x}(t), t) \quad (22)$$

$$D_t^{(-)} f(\mathbf{x}(t), t) = \left(\frac{\partial}{\partial t} + \mathbf{b}_* \cdot \nabla - D\nabla^2 \right) f(\mathbf{x}(t), t) \quad (23)$$

Applying Eq. (22) and Eq. (23) to Eq. (16), combined with Eq. (19) and Eq. (20), one obtains

$$\frac{\partial \vec{u}}{\partial t} = -D\nabla^2 \vec{v} - \nabla(\vec{v} \cdot \vec{u}) \quad (24)$$

$$\frac{\partial \vec{v}}{\partial t} = \mathbf{K} - \vec{v} \cdot \nabla \vec{v} + \vec{u} \cdot \nabla \vec{u} + D\nabla^2 \vec{u} \quad (25)$$

For given initial conditions $\mathbf{x}(t_0)$, $\vec{v}(\mathbf{x}(t_0), t_0)$ and $\vec{u}(\mathbf{x}(t_0), t_0)$, from Eq. (24) and Eq. (25) the current velocity $\vec{v}(\mathbf{x}(t), t)$ and the osmotic velocity $\vec{u}(\mathbf{x}(t), t)$ (or \mathbf{b} and \mathbf{b}_*) can be obtained, then the trajectory $\mathbf{x}(t)$ can be determined by solving Eq. (12). The Markoff process is then completely known.

Let $D = \hbar/(2m)$ and performing the transformation of Eq. (3), Eq. (4), and Eq. (20), one can convert Eq. (24) and Eq. (25) into Eq. (1) and Eq. (2). Combined these two, the Schrodinger equation is resulted. Mathematically, this is just a transformation to convert the two nonlinear differential equations into one linear but complex differential equation.

The derivation of Schrodinger equation based on Eq. (12), Eq. (13) and Eq. (16) is unquestionable. However, it should be noticed that the request of the distinction between forward and backward velocities is very crucial in the above derivation. In contrast, the dynamical equation for classical Brownian motion (Eq. (7)) has $\mathbf{b} = \mathbf{b}_* = \mathbf{K}/\beta$ (see Eq. (8)), which is inconsistent with the requirement of Eq. (12) and Eq. (13). This means Eq. (7) must not be the right dynamical equation for a microscopic system. Consequently, Eq. (16), which was derived from Eq. (7) is unjustified either. Therefore the claim that the derivation is from Newtonian Mechanics or classical Markoff process is ungrounded. The dynamical equation for such a micro-stochastic process, in fact, has not yet been given in the Nelson's approach.

III. Micro-stochastic equation

We now propose a dynamical equation to complete the Nelson's approach named as Micro-Stochastic (MS for short) Equation. We want this dynamical equation in one limit leads to Schrodinger equation when $\beta \rightarrow \infty$, in other limit, reduces to classical mechanics when $\hbar/m \rightarrow 0$. To this end, it must satisfy the following requirements:

$$\lim_{\beta \rightarrow \infty} \text{MS Equation} \Rightarrow \begin{cases} d_{\pm} \mathbf{x}(t) = \mathbf{b}_{\pm}(\mathbf{x}(t), t) dt + d\mathbf{W}_{\pm}(t) & (26) \\ \frac{1}{2} [D_t^{(+)} \mathbf{b}_-(\mathbf{x}(t), t) + D_t^{(-)} \mathbf{b}_+(\mathbf{x}(t), t)] = \mathbf{K}(\mathbf{x}(t)) & (27) \\ D_t^{(+)} \mathbf{b}_+(\mathbf{x}(t), t) - D_t^{(-)} \mathbf{b}_-(\mathbf{x}(t), t) = 0 & (28) \end{cases}$$

$$\lim_{\hbar/m \rightarrow 0} \text{MS Equation} \Rightarrow \frac{d\mathbf{v}(t)}{dt} = \mathbf{K}(\mathbf{x}(t)) \quad (29)$$

Here, for convenience, we have changed the notations, using \mathbf{b}_\pm to denote the forward and backward velocities.

Note that Eq. (26) and Eq. (27) are the same as Eq. (12), Eq. (13) and Eq. (16) in the Nelson's approach, and Eq. (28) is equivalent to Eq. (19). When the transformation of Eq. (3), Eq. (4) and Eq. (19) are made, Eq. (27) and Eq. (28) will lead to Eq. (1) and Eq. (2), thus the Schrödinger equation is resulted. Therefore, if a dynamical equation satisfies the requirement of Eqs. (26)-(28), it will assure the quantum mechanical limit.

The MS equation which satisfies all the above conditions is found to be

$$d_{\mp} \mathbf{v}_{\pm}(t) = -\beta [\mathbf{v}_{\pm}(t) - \mathbf{b}_{\pm}(\mathbf{x}(t), t)] dt + \mathbf{K}_{\pm}(\mathbf{x}(t), t) dt + \beta d\mathbf{W}_{\mp}(t) \quad (30)$$

$$\mathbf{K}_{\pm}(\mathbf{x}(t), t) = \mathbf{K}(\mathbf{x}(t), t) \mp \left(2\vec{u} \nabla \cdot \vec{v} + \frac{\hbar}{m} \nabla^2 \vec{v} \right) \quad (31)$$

$$d_{\pm} \mathbf{x}(t) = \mathbf{v}_{\pm}(t) dt \quad (32)$$

where the notation d_{\pm} denotes the forward (+) and backward (-) individual derivatives without average:

$$d_{+} \mathbf{x}(t) = \lim_{\Delta t \rightarrow 0^{+}} [\mathbf{x}(t + \Delta t) - \mathbf{x}(t)], \quad d_{-} \mathbf{x}(t) = \lim_{\Delta t \rightarrow 0^{+}} [\mathbf{x}(t) - \mathbf{x}(t - \Delta t)] \quad (33)$$

$$d_{+} \mathbf{v}_{\pm}(t) = \lim_{\Delta t \rightarrow 0^{+}} [\mathbf{v}_{\pm}(t + \Delta t) - \mathbf{v}_{\pm}(t)], \quad d_{-} \mathbf{v}_{\pm}(t) = \lim_{\Delta t \rightarrow 0^{+}} [\mathbf{v}_{\pm}(t) - \mathbf{v}_{\pm}(t - \Delta t)] \quad (34)$$

Eq. (26) can be easily proved by using the Nelson's theorem (see reference [6], page 71). Actually the situation is similar to that in the derivation of Eq. (9) case, because when $\beta \rightarrow \infty$, $[\mathbf{K}_{\pm}(\mathbf{x}(t), t)/\beta] \rightarrow 0$. However, we now allow \mathbf{b}_{+} and \mathbf{b}_{-} to be different.

To prove Eq. (27) and Eq. (28), we shall make conditional average to the MS equation Eq. (30). First, we make conditional average at a given $\mathbf{v}(t)$. Knowing that $E_t[d\mathbf{W}_{\mp}(t)] = 0$,

$$\begin{aligned} E_t[d_{-} \mathbf{v}_{+}(t)] &= \lim_{\Delta t \rightarrow 0^{+}} [\mathbf{v}_{+}(t) - \langle \mathbf{v}_{+}(t - \Delta t) \rangle] \\ &= -\beta [\mathbf{v}_{+}(t) - \mathbf{b}_{+}(\mathbf{x}(t), t)] dt + \mathbf{K}_{+}(\mathbf{x}(t), t) dt \end{aligned} \quad (35)$$

$$\begin{aligned} E_t[d_{+} \mathbf{v}_{-}(t)] &= \lim_{\Delta t \rightarrow 0^{+}} [\langle \mathbf{v}_{-}(t + \Delta t) \rangle - \mathbf{v}_{-}(t)] \\ &= \beta [\mathbf{v}_{-}(t) - \mathbf{b}_{-}(\mathbf{x}(t), t)] dt + \mathbf{K}_{-}(\mathbf{x}(t), t) dt \end{aligned} \quad (36)$$

Then we make conditional average at a given $\mathbf{x}(t)$. Knowing that, according to Eq. (26), $E_t[\mathbf{v}_{\pm}(t)] = \mathbf{b}_{\pm}(\mathbf{x}(t), t)$ when $\beta \rightarrow \infty$, we have

$$\lim_{\Delta t \rightarrow 0^{+}} [\mathbf{b}_{+}(\mathbf{x}(t), t) - \langle \mathbf{b}_{+}(\mathbf{x}(t - \Delta t), t - \Delta t) \rangle] = \mathbf{K}_{+}(\mathbf{x}(t), t) dt \quad (37)$$

$$\lim_{\Delta t \rightarrow 0^{+}} [\langle \mathbf{b}_{-}(\mathbf{x}(t + \Delta t), t + \Delta t) \rangle - \mathbf{b}_{-}(\mathbf{x}(t), t)] = \mathbf{K}_{-}(\mathbf{x}(t), t) dt \quad (38)$$

According to the definition of forward and backward derivative (see Eq. (10) and Eq. (11)), we obtain

$$\begin{cases} D_t^{(-)} \mathbf{b}_+(\mathbf{x}(t), t) = \mathbf{K}_+(\mathbf{x}(t), t) \\ D_t^{(+)} \mathbf{b}_-(\mathbf{x}(t), t) = \mathbf{K}_-(\mathbf{x}(t), t) \end{cases} \quad (39)$$

Thus, in order to satisfy Eq. (27), the following constraint is required:

$$\frac{1}{2}[\mathbf{K}_+(\mathbf{x}(t), t) + \mathbf{K}_-(\mathbf{x}(t), t)] = \mathbf{K}(\mathbf{x}(t), t) \quad (40)$$

On the other hand, following Eq. (22) and Eq. (23), it can be shown that

$$D_t^{(+)} - D_t^{(-)} = 2\bar{u}\nabla \cdot + \frac{\hbar}{m}\nabla^2 \quad (41)$$

thus, Eq. (39) can also be written in the following form:

$$\begin{cases} D_t^{(+)} \mathbf{b}_+ - 2\bar{u}\nabla \cdot \mathbf{b}_+ - \frac{\hbar}{m}\nabla^2 \mathbf{b}_+ = \mathbf{K}_+ \\ D_t^{(-)} \mathbf{b}_- + 2\bar{u}\nabla \cdot \mathbf{b}_- + \frac{\hbar}{m}\nabla^2 \mathbf{b}_- = \mathbf{K}_- \end{cases} \quad (42)$$

In order to satisfy Eq. (28), the following constraint is required:

$$\frac{1}{2}[\mathbf{K}_+(\mathbf{x}(t), t) - \mathbf{K}_-(\mathbf{x}(t), t)] = -2\bar{u}\nabla \cdot \vec{v} - \frac{\hbar}{m}\nabla^2 \vec{v} \quad (43)$$

Combined Eq. (40) and Eq. (43) the expressions for \mathbf{K}_\pm (Eq. (31)) are resulted. Thus we have proved that the Schrödinger equation in quantum mechanics is the limit of the MS equation when $\beta \rightarrow \infty$.

It is also important to point out that the momentum operator to be $-i\hbar\nabla$ in quantum mechanics can also be obtained very naturally. In a stochastic process, forward and backward velocities are generally different. Therefore the momentum for a particle with mass m at position \mathbf{x} and time t is naturally defined as the mean value of its forward and backward ones; so is for the kinetic energy. Namely,

$$p(\mathbf{x}, t) = \frac{m\mathbf{b}_+ + m\mathbf{b}_-}{2} = m\vec{v} = \nabla S \quad (44)$$

$$E(\mathbf{x}, t) = \frac{m\mathbf{b}_+^2/2 + m\mathbf{b}_-^2/2}{2} = m v^2/2 + m u^2/2 \quad (45)$$

A straightforward calculation gives

$$p(t) = \int p(\mathbf{x}, t)\rho(\mathbf{x}, t)d^3x = \int \psi^*(\mathbf{x}, t)(-i\hbar\nabla)\psi(\mathbf{x}, t)d^3x \quad (46)$$

$$\overline{E(t)} = \int E(\mathbf{x}, t) \rho(\mathbf{x}, t) d^3x = \int \psi^*(\mathbf{x}, t) \frac{(-i\hbar\nabla)^2}{2m} \psi(\mathbf{x}, t) d^3x \quad (47)$$

This explains why the momentum operator in quantum mechanics is $-i\hbar\nabla$. However, one should note that the correspondence of Eq. (46) is not unique. On the right hand side of Eq. (46), $p(\mathbf{x}, t)$ can be replaced by $m\mathbf{b}_\pm$. This is because the expectation value of \vec{u} is always zero, which can be shown straightforwardly from Eq. (20). Therefore, \mathbf{b}_\pm and \vec{v} give the same expectation value. In quantum mechanics, no such details can be considered.

The classical limit is easy to see. When $\hbar/m \rightarrow 0$, the random force in the MS equation Eq. (30) tends to zero. Consequently, the difference between forward and backward velocities will disappear, namely $\mathbf{v}_+ = \mathbf{v}_- = \mathbf{v}$, and $\mathbf{b}_+ = \mathbf{b}_- = \mathbf{b}$, since they become differentiable. From Eqs. (30)-(32) one immediately sees that $d\mathbf{x}/dt = \mathbf{v} = \mathbf{b}$, and the Newtonian equation $d\mathbf{v}/dt = \mathbf{K}$ is resulted.

IV. Examples and discussions

To convince that the $\beta \rightarrow \infty$ limit of MS equation is indeed equivalent to the quantum mechanics, let us look at some simple examples. Similar to the Nelson's approach, in order to solve the MS equation, one has to first obtain $\mathbf{b}_\pm(\mathbf{x}, t)$ from the asymptotic equations Eq. (27) and Eq. (28) (or $\vec{v}(\mathbf{x}, t)$ and $\vec{u}(\mathbf{x}, t)$ from Eq. (24) and Eq. (25)). There are trivial solutions for these equations. The simplest one is $\vec{v} = k$ (constant), and $\vec{u} = 0$ (or $\mathbf{b}_+ = \mathbf{b}_- = k$) for free motion ($\mathbf{K} = 0$). This in quantum mechanics corresponds to a plane wave $\psi = 1/\sqrt{2\pi} \exp(-ikx)$ in the coordinate space, or a two dimensional rotor, $\psi = 1/\sqrt{2\pi} \exp(ik\varphi)$, in the polar angle space (hereafter we shall use $\hbar = m = 1$ units, and for rotor case m means moment of inertia, and k means angular momentum). Another simple case for free motion is $\vec{v} = 0$ ($\mathbf{b}_+ = -\mathbf{b}_-$), and $\vec{u} = f(\varphi)$ or $f(x)$, which corresponds to a standing wave in quantum mechanics. Having \mathbf{b}_\pm , one can then solve the MS equation to obtain trajectories for each particle in an ensemble.

Two examples are shown in Fig. 1. One is the rotor case with angular momentum $k=1$ (Fig. 1a), the other is the standing wave case with $\vec{v} = 0$ and $\vec{u} = -\tan(\varphi)$, corresponding to the mixing of $k = \pm 1$ wave (Fig. 1b), i. e. $\psi = 1/\sqrt{4\pi} [\exp(ik\varphi) + \exp(-ik\varphi)]$. We see that the probability distributions are well reproduced by the MS approach in both the cases. The curves are calculated by quantum mechanics through the wave functions; the shadow-like fluctuated lines are calculated from the statistics of 10^5 trajectories obtained by directly solving the MS equation (Eq. (30)).

We have also compared the results with that obtained from Eq. (26). They are essentially the same when time is long enough ($t/\tau \gg 1$, $\tau = 1/\beta$ is taken as the time unit in the calculations). This serves as a numerical check to see if Eq. (26) is indeed the asymptotic equation of Eq. (30), when $\beta \rightarrow \infty$.

It is worth to know that the asymptotic solution of the MS equation (Eq. (30)) at $\beta \rightarrow \infty$ is determined entirely by the velocity fields $\mathbf{b}_\pm(\mathbf{x}, t)$ (or $\vec{v}(\mathbf{x}, t)$ and $\vec{u}(\mathbf{x}, t)$), which depend on physical conditions ($\mathbf{K}(\mathbf{x}, t)$) and the initial velocity field of \mathbf{b}_\pm . It does not depend on the initial conditions of individual events. Once $\mathbf{b}_\pm(\mathbf{x}, t)$ is given, no matter what the initial position and momentum for an individual particle put in to solve Eq. (30), they all go to the same steady state guided by the $\mathbf{b}_\pm(\mathbf{x}, t)$ field when $t/\tau \gg 1$. This is understandable, because after long time interaction by the random force, particles may

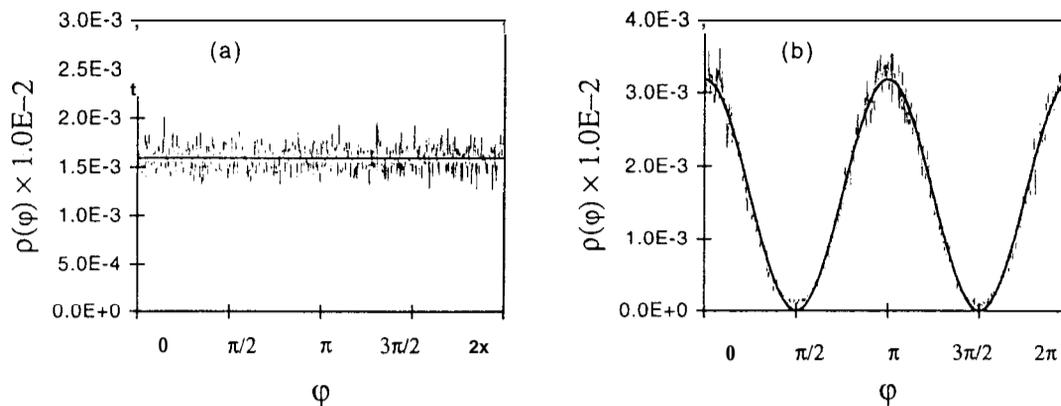


FIG. 1. The probability distributions for (a) the $k = 1$ angular momentum eigen state of a two dimensional rotor, $\psi(\varphi) = (1/\sqrt{2\pi}) \exp(i\varphi)$, and (b) a standing wave state, $\psi(\varphi) = (1/\sqrt{4\pi}) [\exp(i\varphi) + \exp(-i\varphi)]$. The smooth solid lines are obtained from quantum mechanics, and the shadow-like fluctuated lines are calculated from the statistics of 10^5 trajectories at $t/\tau = 1000$ obtained from solving the MS equation.

forget their history completely. In fact, if the fluctuation caused by the random force is so strong for a microscopic system, it is more meaningful to define a statistical initial condition, i.e. the initial probability distributions of the position and momentum of a system, which determine the initial velocity field of \mathbf{b}_{\pm} according to Eq. (19) and Eq. (20), rather than a definite individual initial condition for a individual particle. The later makes sense only when the fluctuation is negligible as that in a macroscopic system.

Having the trajectories obtained from Eq. (30), by statistics one can also calculate the mean values of observables like momenta, energies, etc.. They all agree with what the quantum mechanics obtained. However, the probability distribution of angular momentum deserves special attention. Note that in a stochastic process, there are various definition of velocities: $\mathbf{v}^*(t)$, \mathbf{b}_{\pm} , $\vec{v}(\mathbf{x}, t)$, and $\vec{u}(\mathbf{x}, t)$. Only the current velocity $\vec{v}(\mathbf{x}, t)$ and osmotic velocity $\vec{u}(\mathbf{x}, t)$ that relate to the momentum operator in quantum mechanics as shown in Eqs. (44)-(47). But they are not the real velocity actually carried by each individual particle in the ensemble along its trajectory. The real velocity, kinetic energy, and other related quantities for an individual particle are $\mathbf{v}_+(t)$, $\mathbf{v}_+(t)^2/2m$, etc. .

Two different probability distributions of angular momenta are shown in Fig. 2a and Fig. 2b for the $k = 1$ rotor case. Fig. 2a is for the probability distribution of individual angular momentum \mathbf{v}_+ ; Fig. 2b is for the mean forward angular momentum \mathbf{b}_+ . In this case, $\mathbf{b}_t = m\vec{v} = 1$. Indeed, it has a sharp pick at $k = 1$, which is in consistent with quantum mechanics that $\psi = 1/\sqrt{2\pi} \exp(i\varphi)$ is an eigenfunction of angular momentum \mathbf{L}_z with eigen value $k = 1$. Calculations have been done for other k values, the situation is the same, except the picks appear at different k values. This indicates that the physical meaning of eigen values in quantum mechanics is the conditional expectation value of the angular momentum, and not the real angular momentum for individual particles as have mentioned. The probability distribution of the real angular momentum \mathbf{v}_+ is shown in Fig. 2a, which behaves differently. Although the position of the pick and the mean value remain

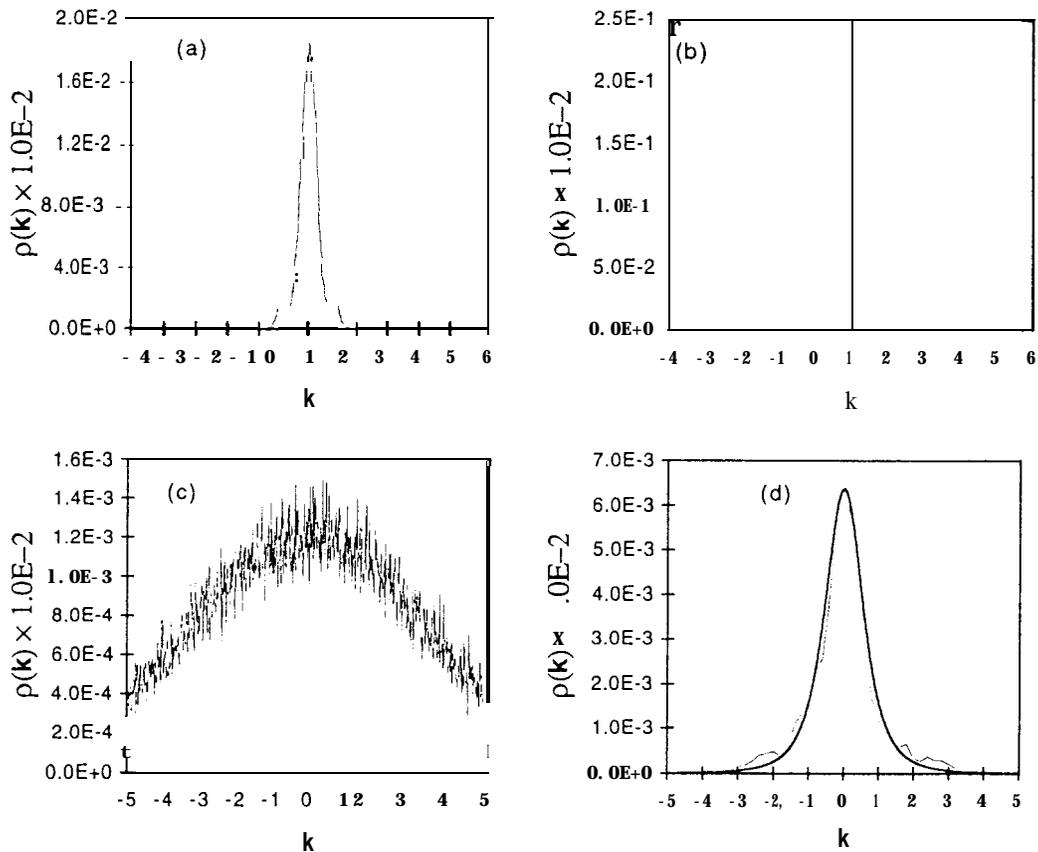


FIG. 2. The angular momentum distributions for the $k = 1$ eigen state [Fig. (a) and Fig. (b)] and the standing wave state [Fig. (c) and Fig. (d)]. In both cases, figures on the left hand side [Fig. (a) and Fig. (c)] are the distributions for the individual angular momentum v_+ ; figures on the right hand side [Fig. (b) and Fig. (d)] are the distributions for the mean forward angular momentum b_+ . The smooth solid lines are obtained from quantum mechanics, and the shadow-like fluctuated lines are calculated from the statistics of 10^6 trajectories at $t/\tau = 1000$ obtained from solving the MS equation.

the same, but the distribution is much wider. This suggests that for an angular momentum eigenstate only the conditional expectation value of the angular momentum that is fixed, the angular momenta of individual particles in the ensemble are varying. The angular momentum conservation only true in the average sense. Individually, it is changing all the time, because the system is not a closed system anymore. According to the stochastic hypothesis, it interacts with the background field randomly. This picture should hold true for other eigen state problems as well.

For the standing wave problem, which is a mixed state of $k = 1$ and $k = -1$ eigen states with equal probabilities, one may expect to see an angular momentum distribution of two peaks appearing at $k = \pm 1$, since angular momentum is quantized in quantum mechanics. It turns out that it is not the case. The distributions are shown in Fig. 2c and

Fig. 2d far b_+ and v_+ , respectively. There is only one pick in both cases appearing at $k=0$, which is the mean value of the angular momentum in this case. The calculations show that particles in the ensemble actually could move with all possible values of angular momenta.

There is a puzzle: Why is the angular momentum not quantized in the micro-stochastic (MS) approach? To solve this puzzle let us go back to review why the angular momentum should be quantized in quantum mechanics? In quantum mechanics, as one of the basic assumptions, it is to assume that only the eigen values of a physical quantity that can be practically measured. According to this assumption of quantum measurement, if a particle is in a state $\psi(\varphi)$, expanding the wave function in terms of eigen functions of L_z , we have

$$\psi_k(\varphi) = (1/\sqrt{2\pi}) \exp(ik\varphi), \quad \psi(\varphi) = \sum C_k \psi_k(\varphi) \quad (48)$$

the right hand side of Eq. (46) and Eq. (47) can then be expressed as

$$p(t) = \sum_k |C_k|^2 k, \quad E(t) = \sum_k |C_k|^2 \frac{k^2}{2m} \quad (49)$$

Since k in the expansion can take only the integer values due to the single valued requirement of $\psi(\varphi)$, Eq. (49) was then interpreted as the angular momentum being quantized, and $|C_k|^2$ is the corresponding probability of finding the angular momentum to be k . This is why one would expect two sharp peaks at $k = \pm 1$ with equal probabilities in the example of the standing wave problem that we have just mentioned.

However, one should note that the original meaning of $|C_k|^2$ is the probability of finding particle in the state $\psi_k(\varphi)$ when the state $\psi(\varphi)$ of a pure ensemble is forced to decompose into a mixed ensemble consisted of $\psi_k(\varphi)$'s. Only under the condition that the angular momentum measurement is applied to the mixed ensemble, $|C_k|^2$ can be interpreted as the probability of finding angular momentum to be k . Therefore, the actual meaning of Eq. (48) and Eq. (49) should be understood as that if a state of a pure ensemble is decomposed into a mixed ensemble of $\psi_k(\varphi)$, then the values of angular momentum can be found in this mixed ensemble is k with probability $|C_k|^2$. Because the series of eigen states for L_z is discrete, only discrete values of k can be measured. Therefore we say the angular momentum is quantized.

From this analysis, we see that the concept of quantization is closely related to the fact that quantum measurements are always carried out by decomposing a pure ensemble into a mixed ensemble consisted of eigen states of the physical quantities that we want to measure. The basic assumption about the quantum measurement in quantum mechanics is true only because this is the way we do the measurements for a microscopic system. Of course, that is probably the best we can do for the quantum measurements at the moment, and the quantum mechanical interpretation is consistent with all the informations provided this way. Nevertheless, it should be noted that the informations provided by these kind of quantum measurements are the informations of the decomposed mixed ensemble and not that of the original pure ensemble. The only information about the original state can be obtained from these measurements is the mean value of the quantity, nothing else.

There is no logical reason to believe that quantization of a physical quantity appearing in a decomposed mixed ensemble, must also appear in the original state. Actually, as what is done in the MS approach, directly calculated the mean values from the left hand side of Eq. (46) and Eq. (47) without doing the wave function expansion, which physically means a direct measurement to obtain $p(\mathbf{x}, t)$, $E(\mathbf{x}, t)$ and $\rho(\mathbf{x}, t)$ without decomposing the state, the same mean values are obtained with no quantization required. Although such direct measurements, which require the measurements for individual events, seem impractical at the moment, conceptually it suggest that the concept of quantization is not fundamentally necessary. In fact, since the left hand side of Eq. (46) and Eq. (47), which derived from the MS approach, has been proved to be equal to the right hand side, which is the standard quantum mechanical formalism, one should be able to obtain the same angular momentum and energy distributions from the pure quantum mechanical approach. Indeed, just do not do the wavefunction decomposition, but recognize that $b(\varphi) = -\tan(\varphi)$, $E(\varphi) = \tan^2(\varphi)/2$ and $\rho = (1/\pi)\cos^2\varphi$, which can be obtained from the wavefunction quantum-mechanically, one obtains

$$\bar{b} = \int_{-\infty}^{\infty} b f_b(b) db, \quad f_b(b) = \frac{2}{\pi} \left(\frac{1}{1+b^2} \right)^2 \quad (50)$$

$$\bar{E} = \int_0^{\infty} E f_E(E) dE, \quad f_E(E) = \frac{4}{\pi} \sqrt{\frac{1}{2E}} \left(\frac{1}{1+2E} \right)^2 \quad (51)$$

where $f_b(b)$ and $f_E(E)$ are the probability density functions for the mean forward angular momentum and energy. In this treatment, even within the framework of quantum mechanics, one sees no quantization at all! Continuous distributions for both angular momentum and energy obtained from the quantum mechanics agree completely with that obtained from the MS approach (see the comparison in Fig. 2d).

V. Summary

In this paper we have followed Nelson's idea to describe the motion of a microscopic object as a stochastic process. However, we found that the dynamics for a microscopic object is fundamentally different from the classical stochastic process. A new dynamical equation named as Micro-Stochastic (MS) equation is proposed. It has been shown that, in one limit, the MS equation leads to the Schrödinger equation in quantum mechanics, when the friction coefficient $\beta \rightarrow \infty$. In other limit, it reduces to Newtonian equation, when $\hbar/m \rightarrow 0$. A two dimensional rotor is taken as a testing example to convince the equivalence between quantum mechanics and the $\beta \rightarrow \infty$ limit of proposed MS approach numerically. Results are positive.

From the comparison with quantum mechanics, a new understanding about the quantization is also proposed. We found that quantization is not the nature of a physical quantity itself but a consequence of quantum measurements, which are based on decomposing a state of a pure ensemble into a mixed ensemble. Introducing the concept of quantization is a convenient way to interpret the information provided by the present quantum measurements, but not fundamentally necessary. In the MS approach, no physical quantities that is quantized; they can take any possible values just like that in the classical case. It has

been shown, at least in one example, that even within the framework of quantum mechanics, just employed some concepts from the MS approach about how to define a physical quantity in a stochastic process, interpreted the results differently, no quantization can be concluded.

It should be pointed out that there is a fundamental constant in the MS equation which has not yet been determined. That is the friction coefficient β . Similar to Einstein's formula for the Brownian motion [6] one can obtain $\beta = 4\epsilon/\hbar$, where ϵ is the average value of energy exchange with the back ground field. In the Brownian motion, ϵ is taken as $kT/2$, since Brownian particles are in thermoequilibrium with surroundings through the collisions with molecules. Here, it should depend on the nature of the interaction with the back ground field. Since we do not want to address this deeper question, we would rather treat it as an unknown constant and hope it can be determined experimentally. However, in both quantum mechanical and classical limits, the results are independent of β . This is good and bad. The good side is that we do not have to know it in the applications within these two limits. The bad side is that we can not get any information about this constant. In addition, if the MS approach can only apply to these two limits then what can we gain from this new approach? We may be able to get better understanding about quantum mechanics conceptually; we may know better about the quantum-classical correspondence; we may use the MS approach to study quantum chaos as well, since by this approach one can obtain trajectories for a microscopic system. However, practically it is still more convenient to use quantum mechanics and classical mechanics in each limit. Nevertheless, there might be a chance to see new physics, if ϵ (or β) is not an universal constant but system dependent. For instance, suppose it is proportional to $1/m$, then for a mesoscopic system, in which β is not large enough to be regarded as infinity and \hbar/m can not be ignored either, the system would definitely behave neither like quantum mechanics, nor the classical mechanics. That would be the most interesting situation.

Finally it should be stressed that many important aspects of quantum mechanics, like spin, identical particles, transition and scattering problems, relativity, quantum field, etc., have not yet been touched in this paper. Therefore, it is too early to draw any firm conclusions at the moment. More testing and comparisons with quantum mechanics will be published elsewhere, and much more studies need to be done.

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References

- [1] D. Bohm and B. J. Hiley, *The Undivided Universe* (Routledge, London, 1993), and references therein.
- [2] E. Nelson, *Quantum Fluctuations*, Princeton Series in Physics (Princeton University Press,

- 1985), and references therein.
- [3] D. Bohm, Phys. Rev. 85, 166 (1952); 85, 180 (1952).
 - [4] E. Nelson, Phys. Rev. 150, 1079 (1966).
 - [5] H. Grabert, P. Hänggi, and P. Talker, Phys. Rev. **A19**, 2440 (1979); D. T. Gillespie, Phys. Rev. A49, 2440 (1994); P. Garbaczewski, and R. Cikiewicz, Phys. Rev. A51, 3445 (1995).
 - [6] E. Nelson, Dynamical Theories of *Brownian* Motion (Princeton University Press, 1967) and references therein.