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KdV Equation as an Euler-Poincaré Equation

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Poincaré had generalised the Euler equation to any Lie algebra. We apply the same method to the infinite dimensional Virasoro algebra to obtain the KdV equation. We have discussed in a quite self-contained way the symmetries of the KdV equation and the Virasoro algebra and their interconnection.

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I. Introduction

The KdV equation is a most remarkable nonlinear partial differential equation in 1+1 dimensions. It was first derived [1] as an evolution equation governing one dimensional, small amplitude, long surface gravity waves propagating in a shallow channel of water. It is the simplest equation we can envisage which incorporates both nonlinearity and dispersion. Subsequently the KdV equation has arisen in a number of other physical contexts, collision-free hydromagnetic waves, stratified internal waves, ion-acoustic waves, plasma physics, lattice dynamics, etc. Although solitary-wave solutions for the KdV equation were discovered long ago, the full solution by the inverse scattering method [2] was available only lately. The inverse scattering method [2] opened up a new arena of the physics of solitons, and many other 1+1 evolution equations like the Sine-Gordon equation, the nonlinear Schrödinger equation, etc. are found to be soluble by this method. The KP equation [3] is an extension of the KdV equation to 2+1 dimensions.

Another surprise came when Gervais [4] showed that the generators of the Virasoro algebra admit a Poisson bracket which can be used to obtain the KdV equation from a hamiltonian. The upshot of this is that the KdV equation is related to the symmetry of the Virasoro algebra, the Lie algebra of the group *diff* S^1 of diffeomorphisms of the circle with central extension. This built a link between integrable systems with 2d conformal field theory. Elegant results in these directions culminated in the Drinfeld-Sokolov hamiltonian reduction theory [5, 6].

The Drinfeld-Sokolov theory is hamiltonian in nature. It gives the KdV equation as a hamiltonian equation of motion in the phase space of the coadjoint orbits of the Virasoro group. Another approach is feasible. It is possible to view the KdV equation as a variational equation of a Lagrangian [7, 8]. This is an analogue of the case of the Euler-Poincaré equation for the finite Lie group $SO(3)$. In the literature [7, 8] the equations are derived, couched in the language of coadjoint action and a direct substitution of equations. This paper attempts to derive everything from first principles emphasising the variational aspects of the problem.

This paper is organised as follows. Section 2 is devoted to a review of the derivation of the classical Euler-Poincard equation because usual textbook does not present an account of this. Section 3 gives a discussion of the KdV equation as a Lax pair and its relation with the Hill's equation. Here we fully discuss the symmetry kinematically. Section 4 is a derivation of the KdV equation as an Euler-Poincare equation. In the last section we present our conclusion and some discussion.

II. The Euler-Poincaré equation

The usual Lagrangian equations do not fully utilise the underlying symmetry of the configuration space. It is possible to formulate a variational principle that suits the symmetry of a Lie group. Poincaré found this variational principle for arbitrary Lie group. For the $SO(3)$ case, this gives the well-known Euler equation for a free top. Hence the resulting equation is called the Euler-Poincard equation. Our presentation below follows that of Arnold [9].

Let us consider a n-dimensional configuration space with coordinates $q^i, i = 1, \ldots, n$. Let $v_1, \ldots, v_n$ be vector fields linearly independent at each point of the n-dimensional manifold. As usual, the commutators of the vector fields are defined as

$$[v_i, v_j] = c_{ij}^k(q) v_k, \tag{1}$$

where

$$c_{ij}^k v_k^m = v_i^l \frac{\partial v_j^m}{\partial q^l} - v_j^l \frac{\partial v_i^m}{\partial q^l} \tag{2}$$

We have adopted the summation over repeated indices convention. We define the quasi-velocities ω^i as the components of the velocities along the vector fields,

$$\dot{q}^i = \omega^l v_l^i. \tag{3}$$

For a variational calculation we consider a δq variation with components δw^i along the vector fields,

$$\delta q^i = \delta w^l v_l^i. \tag{4}$$

Since differentiation with respect to t and variation δ commutes, we get

$$\delta\omega^i = \delta\dot{w}^i + c_{jk}^i \omega^j \delta w^k. \tag{5}$$

Now if we have a Lagrangian $L(q, w)$ depending on q and w, our variational principle will yield the extremal equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \omega^i}\right) = v_i^l \frac{\partial L}{\partial q} - c_{ij}^k \frac{\partial L}{\partial \omega^k}\omega^j .$$

These equations were first obtained by Poincaré in 1901. If we take for v_i the independent vector fields $\partial/\partial q^i$ *we* get back the ordinary Lagrange equations. We can say that these equations are a moving frame generalisation of the Lagrange equations.

Now if the underlying configuration space is a Lie group G we can take $v_1, \ldots, v_n$ to be independent *left-invariant* vector fields on G. In this case c_{ij}^k are constants. Suppose that the Lagrangian L is invariant under left translations on G, that is

$$v_i(L) = v_i^l \frac{\partial L}{\partial q^l} = 0. \tag{7}$$

Hence the Lagrangian L depends on the quasi-velocities ω only. Since the Lie algebra of a group can be realised as left-invariant vector fields on the group, the quasi-velocities ω^i can be regarded as coordinates in the Lie algebra of the group G. In these circumstances the Poincare equations form a closed system of differential equations on the Lie algebra of the group G.

Let us illustrate this with the group SO(3). The Lie algebra generators can be realised as $R^{-1}R$ where R is a group element of SO(3) which is time dependent. These are anti-symmetric matrices and their action on the three-dimensional space can be represented as a vector product, and the quasi-velocities are just the ordinary angular velocities.

$$\dot{\boldsymbol{r}} = \omega \times \boldsymbol{r}, \tag{8}$$

and the variation in position is represented by

$$\delta \boldsymbol{r} = \delta\phi \times \boldsymbol{r}. \tag{9}$$

By the Jacobi identity we get

$$\delta\omega = \delta\dot{\phi} - \omega \times \delta\phi \tag{10}$$

For a free top we can write the Lagrangian as

$$L = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right). \tag{11}$$

We can define the angular momentum $\boldsymbol{M}$ as

$$M^i = \frac{\partial L}{\partial \omega^i}. \tag{12}$$

By variation we can get the Euler-Poincard equation of SO(3) as

$$\dot{\boldsymbol{M}} = \omega \times \boldsymbol{M}. \tag{13}$$

III. KdV and Hill's equation

The KdV equation is well known to have a Lax pair representation for the second order differential operator given by

$$\hat{L} = \partial^2 + u(x,t), \tag{14}$$

where ∂ implies derivative with respect to x. We have the eigenvalue problem

$$(\partial^2 + u)\psi = \lambda\psi. \tag{15}$$

The isospectral flow condition, i.e. the eigenvalues are unchanged with time varying $u(x,t)$ is guaranteed by the condition

$$\partial_t \hat{L} = [\hat{A}, \hat{L}], \tag{16}$$

where $\hat{A}$ is a differential operator. The evolution of the eigenfunctions is then given by

$$\partial_t \psi = \hat{A}\psi. \tag{17}$$

If we require that $\hat{A}$ is a third order differential operator and its commutator with $\hat{L}$ is a pure multiplication operator, this determines $\hat{A}$ as

$$\hat{A} = \partial^3 + \frac{3}{2}u\partial + \frac{3}{4}u', \tag{18}$$

and the evolution equation for $u(x,t)$ is the KdV equation

$$\partial_t u = \frac{1}{4}u''' + \frac{3}{2}u\,u'. \tag{19}$$

We have used the notation $u' \equiv$ du.

Equally we can consider the kernel of the operator $\hat{L}$ which is the Hill's equation and we define the domain of x to be on a circle.

$$(\partial^2 + u)\phi = 0. \tag{20}$$

The above arguments go as the same. We get the same evolution equations for the same Lax pair. Now the Hill's equation admits two basic independent solutions, ϕ_1 and ϕ_2 satisfying

$$(\partial^2 + u)\phi_i = 0, \qquad i = 1, 2. \tag{21}$$

If we make a transformation of coordinates, the Hill's equation will retain its form if the ϕ_i's transforms as differentials of degree $-1/2$. Hence the two basic solutions are not true functions. Rather their ratio

$$s \equiv \left(\frac{\phi_1}{\phi_2}\right) \tag{22}$$

transforms like a scalar function indeed. The Wronskian

$$\phi_1'\phi_2 - \phi_2'\phi_1 \tag{23}$$

is also a true function and it is a constant. We take the normalisation

$$\phi_1'\phi_2 - \phi_2'\phi_1 = 1. \tag{24}$$

Since the Hill's equation is linear, linear combination of solution is also a solution. That is we consider the projective coordinate s as the equivalence class of points of

$$s = \frac{a\phi_1 + b\phi_2}{c\phi_1 + d\phi_2}. \tag{25}$$

It is remarkable that $u(x,t)$ can be expressed in terms of s,

$$2u = \frac{s'''}{s'} - \frac{3}{2}\left(\frac{s''}{s'}\right)^2, \tag{26}$$

where the term appearing on the right hand side of the equation is called the Schwartzian derivative. Indeed, we can introduce two differential operators acting on the function s [10],

$$\theta_1 s = \frac{s''}{s'}, \tag{27}$$

$$\theta_2 s = \frac{s'''}{s'} - \frac{3}{2}\left(\frac{s''}{s'}\right)^2. \tag{28}$$

These two operators are of particular importance for their nice transformation under infinitesimal transformation of the form $\delta s = f s'$,

$$\delta\theta_1 s = f'\theta_1 s + f(\theta_1 s)' + f'', \tag{29}$$

$$\delta\theta_2 s = 2f'\theta_2 s + f(\theta_2 s)' + f'''. \tag{30}$$

This means that $\theta_1 s$ and $\theta_2 s$ transform as differentials of degrees 1 and 2, respectively except for the last anomalous term. We can identify $\theta_1 s$ as b and $\theta_2 s$ as $2u$, and the relation

$$2u = b' - \frac{1}{2}b^2 \tag{31}$$

is the well-known Miura transform in the theory of solitons.

IV. KdV equation and the Virasoro algebra

We now discuss how the KdV equation appears as an Euler-Poincaré equation for the Virasoro group. First of all let us review some preliminary backgrounds for the Virasoro algebra [11]. Virasoro algebra finds its application in two dimensional conformal field theory. It is the basic algebra of the symmetries of conformal field theory. It is associated with the group of diffeomorphisms of the circle S^1 with a central extension. The group action is just the ordinary composition of mappings. The Lie algebra of $diff S^1$ consists of vector fields $g(x)\frac{d}{dx}$ on a circle S^1[12, 13]. The commutation relations are,

$$\left[f\frac{d}{dx}, g\frac{d}{dx}\right] = (fg' - f'g)\frac{d}{dx} + c\int f'g''dx, \tag{32}$$

where c is a parameter signifying the central extension. Dual to the vector field g is the coadjoint vector which is a quadratic differential $u(x)(dx)^2$ and a scalar α dual to c. The pairing between the vector and its dual is

$$\int g(x)u(x)dx - \alpha c. \tag{33}$$

Let us remark that g is a differential of degree -1 and its dual u is a differential of degree 2. Unlike the finite dimensional vector space theory, here the vector and the dual vector are not isomorphic.

Now consider another vector field $f(x)$ acting so that

$$\delta_f x = f, \tag{34}$$

and the variations on g and α can be represented by

$$\delta_f g = fg' - f'g, \tag{35}$$

$$\delta_f \alpha = \int f'g''dx. \tag{36}$$

To keep invariant the inner product we need to have

$$\delta_f c = 0, \tag{37}$$

$$\delta_f u = 2f'u + fu' + cf'''. \tag{38}$$

Notice that u transforms as the Schwartzian derivative. We now take u, f to be time dependent and we take a Lagrangian

$$L = \int u^2(x,t)dx. \tag{39}$$

By power counting, if we required the action to be an invariant we have

$$dt \sim (dx)^3. \tag{40}$$

Suppose we have the variational formula for the scalar function s

$$\delta_f s = fs', \tag{41}$$

it is quite natural to propose the evolution equation for s to be

$$\partial_t s = -\frac{1}{2}us', \tag{42}$$

where we interpret $-\frac{1}{2}u$ in this case as simply the function before s'. The factor before u is taken for convenience so that our notations are consistent. Power counting confirms u is the dual of g. The above equation has also been studied by Wilson [14] identifying u as the Schwartzian derivative and obtaining an evolution equation for s. We do not make such an identification here. Rather we look upon the two equations as the analogous procedures of identifying the component functions of $\partial_t s$ and $\delta_f s$ along s'.

We need to have a second variation giving the formulae,

$$\delta_f \partial_t s = -\frac{1}{2}(\delta_f u)s' - \frac{1}{2}u(f's' + fs''), \tag{43}$$

$$\partial_f \delta_t s = \left(\frac{\partial f}{\partial t}\right) s' - \frac{1}{2}f(u's' + us''). \tag{44}$$

The two variations do not commute but their difference can be evaluated if we take the vector field f to be time independent. Thus we have the relation

$$\delta_f u = -2\left(\frac{\partial f}{\partial t}\right) + (cf''' + u'f + 2uf'). \tag{45}$$

Applying the variation principle to the Lagrangian and after integrating by parts, we get

$$\partial_t u = \frac{c}{2}u''' + \frac{3}{2}uu', \tag{46}$$

which becomes the form of our KdV equation if we choose $c = \frac{1}{2}$.

V. Conclusion

We have demonstrated in a quite self-contained way how to obtain the KdV equation from a Lagrangian using the variational method due to Poincaré applicable on an infinite dimensional Lie group. The symmetry group under consideration is the group of diffeomorphisms of the circle with central extension. The nonlinear term is seen to be related to the scaling dimensions of the "function" u and the dispersion term comes from the central extension of the Lie algebra commutators. We just exploit symmetry arguments to get the evolution equation. This way of argument is tightly related to the underlying symmetry of the configuration space. As is well-known the KdV equation admits an infinite number of conserved constants of motion, it seems possible to use Noether's method or the moment map method to rediscover the constants of motion. These constants of motion are usually obtained by hamitonian method. It is readily seen that our consideration utilises much of the transformation properties of the various variables. Such scaling behaviours are of importance in our discussion. It seems to be a good question to investigate how higher symmetries like the W symmetry can be treated in this manner as we have quite a mature understanding of these higher symmetries via the elegant mathematical edifice of Drinfeld-Sokolov hamitonian reduction.

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References

[1] D. J. Korteweg and G. de Vries, Phil. Mag. (5), 39, 422 (1895).
[2] C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, Phys. Rev. Lett. 19, 1095 (1967).
[3] B. B. Kadomtsev and V. I. Petviashvili, Sov. Phys. Dokl. 15, *539* (1970).
[4] J. L. Gervais, Phys. Lett. 160B, 277 (1985).
[5] V. G. Drinfeld and V. V. Sokolov, J. Sov. Math. 30, 1975 (1985).
[6] A. A. Belavin, Adv. Studies in Pure Math. 19, 117 (1989).
[7] G. Segal, Int. J. Mod. Phys. A6, 2859 (1991).
[8] J. E. Marsden and T. S. Ratiu, *Introduction to Mechanics* and *Symmetry*, TAM (Springer-Verlag, New York, 1994).
[9] *Encyclopaedia of Mathematical Sciences,* V. I. Arnold (Ed.,) Vol. 3, (Springer-Verlag, 1991).
[10] R. C. Gunning, *Lectures* on *Riemann Surfaces,* (Princeton U. Press, New Jersey, 1967).
[11] P. Goddard and D. Olive, Int. J. Mod. Phys. A1, 303 (1986).
[12] E. Witten, Comm. Math. Phys. 114 1 (1988).
[13] N. Hitchin, *Mechanics, Analysis and Geometry* (Elsevier Science Publishers, B.V., 1991), p.359.
[14] G. Wilson, Phys. Lett. A132, 445 (1988).