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Self-Avoiding Loops on the Two-Layer Square Lattice and the Square Lattice with Diagonal Bonds

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We have calculated the series expansions of the generating functions for self-avoiding loops on the two-layer square lattice and the square lattice with diagonal bonds up to 26 and 20 steps respectively by computer. The connective constants, the critical amplitudes and the critical exponents are estimated by the ratio method. Our results indicate that self-avoiding loops on both lattices belong to the same universality class as the ordinary square lattice such that the exponents have the universal value of $\alpha = 0.5$.

PACS. 05.50.+q– Lattice theory and statistics; Ising problems.

I. Introduction

Self-avoiding walks and loops on a lattice are models of linear polymer [1,2]. A self-avoiding walk is a random walk on a lattice such that no site may be visited more than once. A self-avoiding loop is similar to a self-avoiding walk except that the walker returns to the starting point. The generating function of self-avoiding loops with given perimeter on a lattice is usually studied by calculating each term individually by computer, since the closed-form expression is unknown except in special cases. For example, Guttmann and Enting calculated self-avoiding loops to 56 steps on the square lattice [3] and 82 steps on the honeycomb lattice [4]. The generating function of self-avoiding loops on a lattice is given by

$$G(x) = \sum_{n=3}^{\infty} L_n x^n \tag{1}$$

where L_n is the number of self-avoiding loops with perimeter n. The asymtotic form at large n is believed to be [1,5]

$$L_n \approx A\mu^n n^{\alpha-2} \tag{2}$$

where A, μ, and α are called critical amplitude, connective constant, and critical exponent respectively. The critical amplitude and connective constant vary from lattice to lattice. The universality hypothesis predicts that self-avoiding loops on all twodimensional lattices have the same critical exponent of $\alpha = 0.5$ [I], but rigorous proof does not exist. In the

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present paper, we consider the generating functions of self-avoiding loops on the two-layer square lattice and the square lattice with diagonal bonds. The motivation of our study is to estimate the critical exponents in both lattices and then to decide whether the self-avoiding loops on these lattices belong to the same universality class as the square lattice.

II. Two-layer square lattice

The two-layer square lattice consists of two sublattices. Each sublattice is a square lattice such that one sublattice locates on the top of the other and two layers are connected by vertical bonds. Self-avoiding walks on the two-layer square lattice were first studied by us in 1993 [6]. We calculated exactly the number S_n and the mean-square end-to-end distance $(R^2)_n$ of self-avoiding walks with n steps up to 24 steps by computer. The asymptotic forms at large n are [1,5]

$$S_n \approx B\mu^n n^{\gamma-1}, \qquad (R^2)_n \approx Cn^{2\nu}. \tag{3}$$

We used the ratio method of Zinn-Justin [7,8] to estimate the connective constant μ, critical exponents γ, and ν:

$$\mu = 3.553 \pm 0.001, \qquad \gamma = 1.344 \pm 0.004, \qquad \nu = 0.75 \pm 0.01. \tag{4}$$

The conclusion is that self-avoiding walks on the two-layer square lattice belong to the same universality class of two-dimensional lattice where 7 = 43/32 = 1.34375 and ν = 0.75 [1]. The critical amplitudes are not given in our previous paper. Assuming 7 $=43/32, \nu=3/4$, and $\mu=$ 3.553, we estimate the critical amplitudes B and C from the formulae

$$B_n = S_n\mu^{-n}n^{-11/32}, \qquad C_n = (R^2)_n/n^{1.5}, \tag{5}$$

and the results are shown in Table I. Extrapolating the parameters B_n and C_n to infinite n, we get

$$B = 1.144 \pm 0.001, \qquad C = 0.46 \pm 0.01. \tag{6}$$

We have calculated the number of self-avoiding loops up to the 26 steps. Assuming μ = 3.553, we estimate the critical exponent α and critical amplitude A from the formulae

$$\alpha_n -2 + \ln(L_{n+2}/L_n\mu^2)/\ln(1 + 2/n), \qquad A, = L_n\mu^{-n}n^{2-\alpha_n}, \tag{7}$$

and the results are shown in Table II. Extrapolating the parameters to infinite n, we get

$$\alpha = \mathbf{0.50} \pm 0.01, \qquad A = \mathbf{1.07 \pm 0.01}. \tag{8}$$

III. Square lattice with diagonal bonds

We consider a square lattice with diagonal bonds as shown in Fig. 1. The number L_n of self-avoiding loops of n steps on a two-dimensional infinite lattice is related to the number P_n of self-avoiding polygons by

TABLE I. Estimation of the critical amplitudes B and C.

n	B_n	C_n
1	1.40726	1 .00000
2	1.24842	0.84853
3	1.22263	0.73131
4	1.18450	0.67763
5	1.18630	0.62610
6	1.16742	0.60242
7	1.17078	0.57649
8	1.15999	0.56427
9	1.16309	0.54927
10	1.15604	0.54213
11	1.15845	0.53248
12	1.15354	0.52778
13	1.15544	0.52103
14	1.15185	0.51767
15	1.15337	0.51268
16	1.15064	0.51016
17	1.15188	0.50631
18	1.14974	0.50435
19	1.15078	0.50130
20	1.14906	0.49974
21	1.14993	0.49727
22	1.14852	0.49600
23	1.14928	0.49396
24	1.14810	0.49290

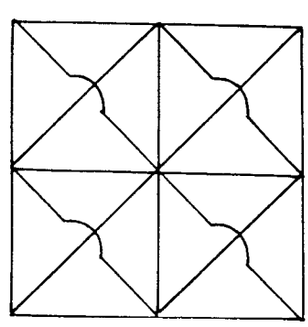

FIG. 1. The square lattice with diagonal bonds

TABLE II. Number of self-avoiding loops on the two-layer square lattice and estimation of the critical exponent α and critical amplitude A.

n	L_n	α_n	A_n
4	16	0.95091	0.42989
6	132	0.66826	0.71334
8	1136	0.62008	0.78851
10	10540	0.47859	1.09219
12	100824	0.44555	1.18564
14	1001588	0.44139	1.19874
16	10268160	0.44514	1.18633
18	107931492	0.44942	1.17175
20	1157145280	0.45252	1.16093
22	12604447108	0.45490	1.15241
24	139099991712	0.45692	1.14503
26	1551946947668		

$$L_n = 2nP_n, \tag{9}$$

if two polygons differ by translation only are considered to be the same. To see this, note that a walker can start his walk at any one of the n vertices along the polygon in either forward or backward direction. It follows from Eqs. (2) and (9) that the asymtotic form of P_n is

$$P_n \approx D\mu^n n^{\alpha-3}. \tag{10}$$

We have calculated the number of self-avoiding polygons on the square lattice with diagonal bonds up to 20 steps. We define D_n, μ_n, and α_n by the following equations:

$$\begin{aligned} P_{n+1} &= D_n\mu_n^{n+1}(n+1)^{\alpha_n-3} \\ P_n &= D_n\mu_n^n n^{\alpha_n-3}, \\ P_{n-1} &= D_n\mu_n^{n-1}(n-1)^{\alpha_n-3}. \end{aligned} \tag{11}$$

It is easy to see that for very large n, D_n, μ_n and α_n converge to D, μ, and α respectively. The solution of Eq. (11) is

$$\begin{aligned} \alpha_n &= 3 + \ln(P_{n+1}P_{n-1}/P_n^2)/\ln(1-1/n^2), \\ \mu_n &= (1+1/n)^{3-\alpha_n}P_{n+1}/P_n, \\ D_n &= P_n\mu_n^{-n}n^{3-\alpha_n}. \end{aligned} \tag{12}$$

TABLE III. Number of self-avoiding polygons on the square lattice with diagonal bonds and estimation of the critical exponent α, critical amplitude D and connective constant μ.

n	P_n	α_n	D_n	μ_n
3	4			
4	12			
5	36			
6	126			
7	476			
8	1941			
9	8272			
10	36638	0.24708	0.39262	5.91954
11	166828	0.18545	0.42667	5.95441
12	777589	0.30074	0.36137	5.89498
13	3693180	0.29990	0.36183	5.89538
14	17824204	0.34084	0.33887	5.87752
15	87202116	0.34974	0.33386	5.87391
16	431688854	0.36925	0.32272	5.86652
17	2159167600	0.37936	0.31689	5.86252
18	10898001788	0.39127	0.30994	5.85893
19	55451025168	0.40006	0.30476	5.85615
20	284187147131			

The number of polygons and the numerical values of α_n, D_n, and μ_n are given in Table III. Extrapolating the parameters to infinite n, we obtain

$$\alpha = 0.50 \pm 0.02, \qquad D = 0.26 \pm 0.01, \qquad \mu = 5.83 \pm 0.01. \tag{13}$$

IV. Conclusion

We considered the self-avoiding walks and loops on the two-layer square lattice and estimated the critical exponent α and the critical amplitudes A, B, and C. The results are given in Eqs. (6) and (8). We calculated the number of self-avoiding polygons on the square lattice with diagonal bonds up to the 20 steps and estimated the critical exponent α, critical amplitude D and connective constant μ which are shown in Eq. (13). The estimated value of critical exponent α is 0.5 for both lattices. Our results indicate that self-avoiding loops on the two-layer square lattice and the square lattice with diagonal bonds have the same critical exponent α as the self-avoiding loops on ordinary two-dimensional lattices such as square lattice, triangular lattice, etc.

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