

## Correlation Duality Relations for the $(N_\alpha, N_\beta)$ Model

F. Y. Wu and Wentao T. Lu

*Department of Physics, Northeastern University,  
Boston, Massachusetts 02115, U.S.A.*

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Duality relations for the correlation functions of  $n$  sites on the boundary of a planar lattice are derived for the  $(N_\alpha, N_\beta)$  model of Domany and Riedel for  $n = 2, 3$ . Our result holds for arbitrary lattices which can have nonuniform interactions.

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### I. Introduction

The duality relation for the Potts model [1,2] is an identity [3] relating the partition functions of a Potts model on a planar lattice with that of its dual. Very recently, one of us [4] has extended the duality consideration to Potts correlation functions. Specifically, it was established that certain duality relations exist for correlation functions of any  $n$  Potts spins sitting on the boundary of a planar graph. Explicit expressions for the duality relation were then obtained for  $n = 2, 3$ . While for  $n = 2$  the relation generalizes a known expression [5,6] for the 2-point correlation function of the Ising model, the general  $n$  results are new. It was also stated in [4] that the formulation can be extended in a straightforward fashion to  $n \geq 4$ , and to the more general  $(N_\alpha, N_\beta)$  model of Domany and Riedel [7].

Jacobsen [8] has since pointed out that the procedure as described in [4] does not extend straightforwardly to  $n \geq 4$ . But Wu and Huang [9] subsequently showed that the procedure can be made to work, provided that one invokes certain correlation identities which have not previously been known. In this paper we address the other extension of the formulation mentioned in [4], namely, the extension to the  $(N_\alpha, N_\beta)$  model. Here, we report results for  $n = 2, 3$ .

### II. The $(N_\alpha, N_\beta)$ model

The  $(N_\alpha, N_\beta)$  model [7] is an  $N_\alpha N_\beta$ -state spin model; where  $N_\alpha$  and  $N_\beta$  are positive integers. For brevity we shall rename

$$q \equiv N_\alpha, \quad q' \equiv N_\beta. \quad (1)$$

The model is most conveniently visualized as being represented by placing two Potts spins at each site. At each site of a lattice, or more generally a graph,  $G$  of  $N$  sites, one places two Potts spins of  $q$  and  $q'$  states: respectively. Denote the spins states by  $\sigma = 0, 1, \dots, q-1$  and  $\tau = 0, 1, \dots, q'-1$ . Then, the interaction energy  $E(\sigma, \tau; \sigma', \tau')$  between two sites in spin states  $\sigma, \tau$  and  $\sigma', \tau'$  is given by

$$\begin{aligned}
 -E(\sigma, \tau; \sigma', \tau')/kT &= K_{00}\delta_{\sigma, \sigma'}\delta_{\tau, \tau'} + K_{01}\delta_{\sigma, \sigma'}(1 - \delta_{\tau, \tau'}) \\
 &+ K_{10}(1 - \delta_{\sigma, \sigma'})\delta_{\tau, \tau'} + K_{11}(1 - \delta_{\sigma, \sigma'})(1 - \delta_{\tau, \tau'}),
 \end{aligned}
 \tag{2}$$

where  $k$  is the Boltzmann constant,  $T$  the temperature, and  $\delta$  the Kronecker delta function. The  $(N_\alpha, N_\beta)$  model becomes the Ashkin-Teller model [10] when  $N_\alpha = N_\beta = 2$ .

The partition function of the  $(N_\alpha, N_\beta)$  model is given by

$$Z = \sum_{\sigma_i} \sum_{\tau_i} \prod_{\text{edges}} \exp[-E(\sigma_i, \tau_i; \sigma_j, \tau_j)/kT],
 \tag{3}$$

where the product is taken over all pairs connected by edges in  $G$ .

Consider  $n$  sites on the boundary of  $\mathcal{L}$ . The probability that the  $n$  sites will be in specific definite states  $\{\sigma_1, \tau_1\}, \{\sigma_2, \tau_2\}, \dots, \{\sigma_n, \tau_n\}$  is given by

$$P(\sigma_1, \sigma_2, \dots, \sigma_n | \tau_1, \tau_2, \dots, \tau_n) = Z_{\sigma_1, \sigma_2, \dots, \sigma_n; \tau_1, \tau_2, \dots, \tau_n} / Z,
 \tag{4}$$

where  $Z_{\sigma_1, \sigma_2, \dots, \sigma_n; \tau_1, \tau_2, \dots, \tau_n}$  is the partial partition function defined by (3) with the  $n$  sites in definite states. Then, following [4], one can define an  $n$ -point correlation

$$\Gamma_n \equiv (qq')^n P(\sigma, \sigma, \dots, \sigma | \tau, \tau, \dots, \tau) - \mathbf{1},
 \tag{5}$$

where  $P(\sigma, \sigma, \dots, \sigma | \tau, \tau, \dots, \tau)$  is the probability that all  $n$  sites are in the same state. Clearly,  $\Gamma_n$  vanishes identically when there is no correlation between the  $n$  sites.

### III. Duality relation

The partition function (3) possess a duality relation [3, 7]. Let  $Z^*$  be the partition function of an  $(N_\alpha, N_\beta)$  model on the dual of  $G$  which has  $N^*$  sites with interactions also given by (2), but with  $K_{mn}$  replaced by  $K_{mn}^*$ ,  $m, n = 0, 1$ . Further introduce Boltzmann factors  $u_{mn} = e^{K_{mn}}$  and  $u_{mn}^* = e^{K_{mn}^*}$ . Then, it is well-known [3, 7] that the duality relation assumes the form

$$Z(u_{00}, u_{01}, u_{10}, u_{11}) = (qq'C)Z^*(u_{00}^*, u_{01}^*, u_{10}^*, u_{11}^*),
 \tag{6}$$

where  $C = (qq')^{-N^*}$  and

$$\begin{aligned}
 u_{00}^* &= u_{00} + (q - 1)u_{10} + (q' - 1)u_{01} + (q - 1)(q' - 1)u_{11} \\
 u_{01}^* &= u_{00} - u_{01} + (q - 1)(u_{10} - u_{11}) \\
 u_{10}^* &= u_{00} - u_{10} + (q' - 1)(u_{01} - u_{11}) \\
 u_{11}^* &= u_{00} - u_{01} - u_{10} + u_{11}.
 \end{aligned}
 \tag{7}$$

Note that the transformation (7) can be written more compactly as

$$\mathbf{u}^* = \mathbf{T}_2(q) \cdot \mathbf{u} \cdot \tilde{\mathbf{T}}_2(q'),
 \tag{8}$$

where

$$\mathbf{u} = \begin{pmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{pmatrix}, \mathbf{u}^* = \begin{pmatrix} u_{00}^* & u_{01}^* \\ u_{10}^* & u_{11}^* \end{pmatrix}, \mathbf{T}_2(q) = \begin{pmatrix} 1 & q-1 \\ 1 & -1 \end{pmatrix}, \quad (9)$$

and  $\tilde{\mathbf{T}}$  is the transpose of  $\mathbf{T}$ . We shall refer to (6) as the fundamental duality relation which applies to arbitrary graph  $G$  with arbitrary (nonuniform) edge interactions. As discussed in [4], the duality relations for correlation functions are most conveniently obtained by applications of this fundamental duality relation.

#### IV. The 2-point correlation function

In this section we consider the duality for the 2-point correlation functions between any two spins at sites  $i$  and  $j$  on the boundary of  $G$ . Following [4, 9], this is done by applying (6) to an auxiliary graph (lattice). The auxiliary graph is obtained from  $G$  by connecting sites  $i$  and  $j$  with an auxiliary edge as shown in Fig. 1(a). Let  $u_{mn}$  and  $u_{mn}^*$  be the respective Boltzmann factors associated with this edge and its dual related by (7). Applying the fundamental duality to the auxiliary graph of Fig. 1(a), we arrive at (6) in the form

$$Z_{\text{aux}} = C Z_{\text{aux}}^*, \quad (10)$$

where we have used the fact that the dual of the auxiliary graph has  $N^* + 1$  sites, and

$$\begin{aligned} Z_{\text{aux}} &= q q' [u_{00} Z_{00,00} + (q-1) u_{10} Z_{01,00} \\ &\quad + (q'-1) u_{01} Z_{00,01} + (q-1)(q'-1) u_{11} Z_{01,01}] \\ Z_{\text{aux}}^* &= q q' [u_{00}^* Z_{00,00}^* + (q-1) u_{10}^* Z_{01,00}^* \\ &\quad + (q'-1) u_{01}^* Z_{00,01}^* + (q-1)(q'-1) u_{11}^* Z_{01,01}^*]. \end{aligned} \quad (11)$$

Note that in writing down (11) we have made use of the degeneracy  $Z_{00,00} = Z_{\alpha\alpha;\beta\beta}$ ;  $Z_{00,01} = Z_{\alpha\alpha;\alpha\beta}$ ,  $\alpha \neq \beta$ ; etc.

Substituting (7) into (11), the dual relation (10) becomes linear in both  $u_{mn}$  and  $Z_{\sigma_i\sigma_j,\tau_i\tau_j}$ . It is now a simple matter to equate the coefficients of the 4  $u_{mn}$ 's, and obtain

$$\begin{aligned} Z_{00,00} &= C [Z_{00,00}^* + (q-1) Z_{01,00}^* + (q'-1) (Z_{00,01}^* + (q-1) Z_{01,01}^*)] \\ Z_{01,00} &= C [Z_{00,00}^* - Z_{01,00}^* + (q'-1) (Z_{00,01}^* - Z_{01,01}^*)] \\ Z_{00,01} &= C [Z_{00,00}^* - Z_{00,01}^* + (q-1) (Z_{01,00}^* - Z_{01,01}^*)] \\ Z_{01,01} &= C [Z_{00,00}^* - Z_{01,00}^* - Z_{00,01}^* + Z_{01,01}^*]. \end{aligned} \quad (12)$$

This relation can be written more compactly as

$$\mathbf{Z}_2 = C \mathbf{T}_2(q) \cdot \mathbf{Z}_2^* \cdot \tilde{\mathbf{T}}_2(q'), \quad (13)$$

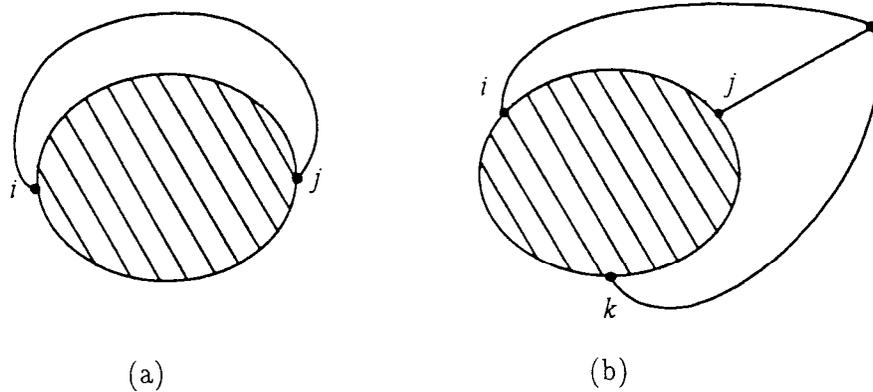


FIG. 1. (a) The auxiliary graph for  $n = 2$ . (b) The auxiliary graph for  $n = 3$ .

where  $\mathbf{Z}_2$  is a  $2 \times 2$  matrix with  $(\beta\beta')$ -th element  $Z_{0\beta;0\beta'}$ , and similarly for  $\mathbf{Z}_2^*$ . This is the desired duality relation for the 2-point correlation function.

To compute the correlation (5) for  $n = 2$ , we use (4) and note that one can apply the fundamental duality relation (6) to  $\mathcal{L}$  to write the partition function  $Z$  in the form of

$$Z = qq' CZ^* = (qq')^2 CZ_{00;00}^*. \tag{14}$$

Substituting (12) and (14) into (5), one is led to the result

$$\begin{aligned} \Gamma_2 &= (4-1)p_{01;00} + (q'-1)p_{00;01} + (4-1)(q'-1)p_{01;01} \\ &= \mathbf{v}_2(q) \cdot \mathbf{P}_2 \cdot \tilde{\mathbf{v}}_2(q') - 1, \end{aligned} \tag{15}$$

where

$$p_{\alpha\beta;\alpha'\beta'} \equiv Z_{\alpha\beta;\alpha'\beta'}^* / Z_{00;00}^*, \quad \alpha, \beta, \alpha', \beta' = 0, \mathbf{I}, \tag{16}$$

with

$$\begin{aligned} \mathbf{v}_2(g) &= (1, g-1), \\ \mathbf{P}_2 &= \left( \frac{1}{Z_{00;00}^*} \right) \mathbf{Z}_2^*. \end{aligned} \tag{17}$$

The expression (16) generalizes the  $q' = 1$  result for the Potts model [4].

### V. The 3-point correlation function

For the 3-point correlation functions, we apply the fundamental duality relation to the auxiliary graph shown in Fig. 1(b), and extract from the resulting expression the desired duality relation. Now for  $n = 3$  there are 25 independent 3-point correlation functions and

the algebra tends to be involved. The procedure of [4] can nevertheless be carried through, and the result can be expressed compactly as suggested by (14) as follows.

Introduce a 5 x 5 matrix  $Z_3$  whose elements are the 3-point correlation functions

$$Z_3 = \begin{pmatrix} Z_{000;000} & Z_{000;001} & Z_{000;010} & Z_{000;100} & Z_{000;012} \\ Z_{001;000} & Z_{001;001} & Z_{001;010} & Z_{001;100} & Z_{001;012} \\ Z_{010;000} & Z_{010;001} & Z_{010;010} & Z_{010;100} & Z_{010;012} \\ Z_{100;000} & Z_{100;001} & Z_{100;010} & Z_{100;100} & Z_{100;012} \\ Z_{012;000} & Z_{012;001} & Z_{012;010} & Z_{012;100} & Z_{012;012} \end{pmatrix}. \quad (18)$$

and similarly for  $Z_3^*$ . Then, we find

$$Z_3 = \left( \frac{C}{qq'} \right) T_3(q) \cdot Z_3^* \cdot \tilde{T}_3(q') \quad (19)$$

where

$$T_3(q) = \begin{pmatrix} 1 & q-1 & q-1 & q-1 & (q-1)(q-2) \\ 1 & -1 & q-1 & -1 & -(q-2) \\ 1 & -1 & -1 & q-1 & -(q-2) \\ 1 & q-1 & -1 & -1 & -(q-2) \\ 1 & -1 & -1 & -1 & 2 \end{pmatrix}. \quad (20)$$

The 3-point correlation (5) can be computed straightforwardly using (19) and the identity

$$Z = (qq')^2 C Z_{000;000}^*. \quad (21)$$

This leads to the compact expression which extends (15),

$$\Gamma_3 = \mathbf{v}_3(q) \cdot \mathbf{p}_3 \cdot \tilde{\mathbf{v}}_3(q') - 1, \quad (22)$$

where the row vector  $\mathbf{v}_3$  and the 5 x 5 matrix  $\mathbf{p}_3$  are given by

$$\mathbf{v}_3(q) = (1, q-1, q-1, q-1, (q-1)(q-2)),$$

$$\mathbf{p}_3 = \left( \frac{1}{Z_{000;000}^*} \right) \mathbf{z}_3. \quad (23)$$

## VI. Summary and acknowledgement

We have obtained the duality relations for the 2- and 3-site boundary correlation functions for the  $(N_\alpha, N_\beta)$  model, which includes the  $N_\alpha = N_\beta = 2$  Ashkin-Teller model

as a special case. Explicit expressions are also obtained for the 2- and 3-point correlations. Our results are presented in compact forms which are suggestive of possible extensions to higher correlations and to the chiral Potts model [11]. These extensions will be reported elsewhere [12].

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