

## Stochastic Hopf Bifurcations

H. K. Leung

*Department of Physics, National Central University  
Chung-li, Taiwan 320, R.O.C.*

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A nonlinear model system of chemical reaction, which exhibits supercritical Hopf bifurcation is studied stochastically by taking white noise into consideration. Both the asymptotic properties near the bifurcation point and the transient processes preceding to the competing attractors are emphasized. It is found that both additive and multiplicative noises tend to suppress periodicity, and that only the additive noises could induce transition from a limit cycle to a fixed point. This finding is in accord with the previous results that Hopf bifurcation is always postponed by noises. Extensive investigation results in a phase diagram showing the phase domains of competing attractors. The phase boundaries are interpreted as the bifurcation loci for the stochastic Hopf bifurcation.

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### I. Introduction

Studies of noise effects in nonlinear systems suggest that noises are no longer associated with the traditional aspects of incoherence and disturbance [1]. Noises are found to enhance the response of dynamic systems to the external periodic driving [2]. Stochastic resonance, a phenomenon resulting from the interplay of randomness, periodicity and nonlinearity, has significant impact in physical, biological and engineering sciences [3]. More recently, noises are utilized to control and synchronize chaos [4].

Even the destructive aspects of noises can be maneuvered to induce transitions among different steady states allowed by the nonequilibrium system [5]. This kind of transition resembles the second order phase transition in equilibrium systems, and therefore enriches our understanding of the phase transition and critical phenomena in general.

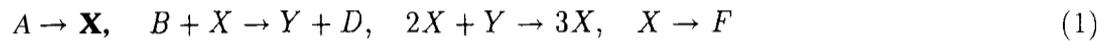
Mechanism of the noise-induced phenomena rests on the interaction between noises and the attractors of the system. Though fixed point and limit cycle are the two simplest and most fundamental attractors in dynamic systems, researches are almost exclusively on the fixed point [1]. Furthermore, transient behaviors are usually neglected and attentions are concentrated mostly on the passage time statistics in multistable systems. Noise studies on limit cycle oscillations are meager since the difficulty arising in stochastic analysis increases nonlinearly when the dimension of the system increases from one to two [6-9].

Noise effects depend on the types of attractor and on the nonlinearity of a given model. Noises weaken stability of attractors to various extents, and therefore could affect the bifurcation process in a system with competing attractors by shifting the deterministic instability. The Brusselator model of chemical reaction, a paradigm of Hopf bifurcation, have been studied stochastically by several authors. Some results suggested that bifurcation from fixed point to limit cycle is always postponed [10,11] by the noisy perturbations, while both advancement and postponement are predicted to be possible in other studies [12]. These conclusions are usually derived by analyzing the steady state properties of the probability function. Stochastic transient processes which play significant role in nonlinear systems [10,13] were neglected in these investigations.

In this report we study the same model system, with emphasize on the transient processes preceding to the asymptotic attractor. Stochastic formulation of various types of white noises will be presented in Sec.3. Time-dependent properties near the deterministic bifurcation point and the stochastic phase diagrams of competing attractors will be discussed in Sec.4 and 5 respectively. In Sec. 6, we shall discuss the transient aspects of the noise-suppressed periodicity and the postponement of Hopf bifurcation.

## II. Deterministic Hopf bifurcation

The Brusselator model describing a series of chemical reactions,



can be characterized by a set of coupled rate equations [6].

$$\dot{x} = f_1(x, y) = A - (1 + B)x + x^2 y \quad (2)$$

$$\dot{y} = f_2(x, y) = Bx - x^2 y \quad (3)$$

In the above,  $(x, y)$  represent the concentrations of chemical species  $(X, Y)$  which are monitored during the reaction process, and are considered as time-dependent variables of the nonlinear system. The parameters  $A$  and  $B$  stand for those which can be adjusted to some constant values through external means, are considered as control parameters of the system.

This famous model is a simple one which allows both fixed point and limit cycle attractors. These two competing attractors are intrinsic to the dynamic system, and exist for different domains in parameter space  $(A, B)$ . The control parameters  $A$  and  $B$  decide the fate of competition of the two attractors. Transition from the one to another occur at the *bifurcation point* specified by proper choices of  $(A, B)$ . This model is a paradigm of Hopf bifurcation in nonlinear dynamics.

The equations (2) and (3) allow a steady solution with

$$(x_s, y_s) = (A, B/A) \quad (4)$$

The reaction system starting from any initial state  $(x_o, y_o)$  will relax asymptotically toward this unique state, provided that the eigenvalues

$$\lambda = \frac{1}{2} [B - 1 - A^2 \pm [(B - 1 - A^2)^2 - 4A^2]^{\frac{1}{2}}] \quad (5)$$

remain negative. In another word, for a given value of  $A$ , the fixed point  $(A, B/A)$  is stable if  $B < B_c$ , where

$$B_c = 1 + A^2 \quad (6)$$

This steady state is a stable one and is an attractor of the system if  $B < B_c$ . As  $B$  approaches  $B_c$  from below, the stability of the fixed point decreases, and the relaxation time increases. At the critical point of  $B = B_c$ , the relaxation time tends to be infinite since the damping of spiral trajectory is infinitely slow. For  $B > B_c$ , the fixed point turns to be unstable and all phase trajectories will relax toward a limit cycle, which has its shape and period determined by  $A$  and  $B$ .

For the sake of convenience, we shall set  $A = 1$  and let  $B$  be the sole *bifurcation* parameter. The point  $B_c = 2$  in the parameter space is the Hopf *bifurcation* point, across which a fixed point and a limit cycle can transform to each other. Deterministically, both advancement (with  $B_c < 2$ ) and postponement (with  $B_c > 2$ ) of bifurcations are not allowed.

### III. Stochastic formulations

Since the reaction mechanism and the parameters  $A$  and  $B$  are related to external modulations, random elements of various types could affect the dynamic process. Within the framework of white noise formulation [5,14], the rate equations (2) and (3) turn to be stochastic ones,

$$\dot{x} = f_1(x, y) + D_1 \xi_{1,t} g_1(x, y) \quad (7)$$

$$\dot{y} = f_2(x, y) + D_2 \xi_{2,t} g_2(x, y), \quad (8)$$

where  $\xi_{i,t}$  are Wiener processes with

$$\langle \xi_{i,t} \rangle = 0, \quad \langle \xi_{i,t} \xi_{j,t'} \rangle = \delta_{ij} \delta(t - t'). \quad (9)$$

This general form represents various types of noises. If the parameter  $A$  is fluctuating or if the the rate equation of  $x$  is subject to an additional random driving, we have,

$$g_1(x, y) = 1, \quad g_2(x, y) = 0. \quad D_1 \neq 0, \quad D_2 = 0. \quad (10)$$

For another type of additive noises, which represents the additional random driving in the rate equation of  $y$ , we have,

$$g_1(x, y) = 0, \quad g_2(x, y) = 1, \quad D_1 = 0, \quad D_2 \neq 0. \quad (11)$$

The third type of additive noise is a combination of the above two, such that  $g_1 = g_2 = 1$  and that  $D_1 = D_2 \neq 0$ . This corresponds to the case for which both production mechanism of  $x$  and  $Y$  are perturbed by the same source of additional driving.

If the bifurcation parameter is fluctuating about its mean  $B$  and with an amplitude  $D_B$ , or  $B_t = B + D_B \xi_t$ , we have,

$$g_1(x, y) = -x, \quad g_2(x, y) = x, \quad D_1 = D_2 = D_B. \quad (12)$$

This represents a multiplicative noise of first order, and appears in both the rate equations of  $x$  and  $y$ .

The order parameters  $x(t)$  and  $y(t)$  are now stochastic variables, and so are described by the probability function  $P(x, y; t)$  which satisfies a Fokker-Planck equation [14].

$$\begin{aligned} \partial_t P = & -\partial_x \left( f_1 + \frac{\nu-1}{2} D_1^2 g_1 \partial_x g_1 \right) P + \frac{1}{2} D_1^2 \partial_{xx} (g_1^2 P) + D_B^2 \partial_{xy} (g_x g_y P) \\ & -\partial_y \left( f_2 + \frac{\nu-1}{2} D_2^2 g_2 \partial_y g_2 \right) P + \frac{1}{2} D_2^2 \partial_{yy} (g_2^2 P) \end{aligned} \quad (13)$$

In the above,  $\nu = 1$  and  $\nu = 2$  stand respectively for the results derived from the Ito and Stratonovich interpretations of stochastic calculi.

An explicit solution of  $P(x, y; t)$  is unlikely. Instead we could employ the Gaussian approximation in such a way that the peak of  $P(x, y; t)$  is located at  $(\bar{x}, \bar{y})$  and that the shape is described by the variances

$$\sigma_{ij} = \overline{x_i x_j} - \bar{x}_i \bar{x}_j \quad (14)$$

where  $x_1 = x, x_2 = y$ , and the statistical moments of  $P(x, y; t)$  are defined by

$$\overline{x_i^m x_j^n} = \int_0^\infty dx_i dx_j x_i^m x_j^n P(x, y; t) \quad (15)$$

The behaviors of the first two moments can be derived with the help of the Fokker-Planck equation (13). It is readily found that,

$$\dot{\bar{x}}_i = \bar{f}_i + \frac{\nu-1}{2} D_i^2 \overline{g_i \partial_{x_i} g_i} \quad (16)$$

$$\begin{aligned} \dot{\sigma}_{ij} = & \overline{x_i f_j} - \bar{x}_i \bar{f}_j + \overline{x_j f_i} - \bar{x}_j \bar{f}_i + D_i^2 \overline{g_i^2 \delta_{ij}} + D_B^2 \overline{g_i g_j \delta_{ix} \delta_{iy}} \\ & + \frac{\nu-1}{2} [D_i^2 (\overline{x_j g_i \partial_{x_i} g_i} - \bar{x}_j \overline{g_i \partial_{x_i} g_i}) + D_j^2 (\overline{x_i g_j \partial_{x_j} g_j} - \bar{x}_i \overline{g_j \partial_{x_j} g_j})] \end{aligned} \quad (17)$$

With a proper truncation scheme [13], these equations can be approximated into a set of five coupled moment equations in a closed form,

$$\dot{\bar{x}} = A - (1 + B)\bar{x} + \bar{x}^2 \bar{y} + \bar{y} \sigma_{xx} + 2\bar{x} \sigma_{xy} - \frac{\nu-1}{2} D_B^2 \bar{x} \quad (18)$$

$$\dot{\bar{y}} = B\bar{x} - \bar{x}^2 \bar{y} - \bar{y} \sigma_{xx} - 2\bar{x} \sigma_{xy} \quad (19)$$

$$\dot{\sigma}_{xx} = [4\bar{x} \bar{y} - 2 - B] \sigma_{xx} + 2\bar{x}^2 \sigma_{xy} + D_1^2 + D_B^2 (\nu \sigma_{xx} + \bar{x}^2) \quad (20)$$

$$\dot{\sigma}_{yy} = -2\bar{x}^2 \sigma_{yy} + 2(B - 2\bar{x} \bar{y}) \sigma_{xy} + D_2^2 + D_B^2 (\sigma_{xx} + \bar{x}^2) \quad (21)$$

$$\dot{\sigma}_{xy} = [2\bar{x} \bar{y} - 1 - B - \bar{x}^2] \sigma_{xy} + \bar{x}^2 \sigma_{yy} + (B - 2\bar{x} \bar{y}) \sigma_{xx} + D_B^2 \left( \frac{\nu-1}{2} \sigma_{xy} - \sigma_{xx} - \bar{x}^2 \right) \quad (22)$$

With a given initial condition of  $(\bar{x}(0), \bar{y}(0)) = (x_o, y_o)$  and  $\sigma_{ij}(0) = 0$ , the probability function  $P(x, y; t)$  can be solved approximately by solving the five moments  $\bar{x}_i(t)$  and  $\sigma_{ij}(t)$  from the above equations.

#### IV. Stochastic transient near bifurcation

For most cases with small and moderate noise intensity, transient processes and asymptotic properties of the system are more or less deterministic. The averaged phase trajectories  $(\bar{x}(t), \bar{y}(t))$  are similar to the noise-free  $(z(t), y(t))$ , while fluctuations and correlations  $\sigma_{ij}(t)$  are small. Therefore the bifurcation processes are not much affected by noises. Deviation from deterministic behaviors occurs only if the system is subject to large noises.

On the fixed point side of the bifurcation,  $B < 2$ , fluctuations remain to be small even if the bifurcation point is approached. This enable us to probe the transient characteristics of the noisy bifurcation. On the other side of the bifurcation point, limit cycle is relatively more vulnerable to the noise effect since that  $\sigma_{ij}(t)$  is oscillating with small amplitudes which is glowing very slowly with time. This situation is similar to other noisy oscillating systems [6-9]. As in the traditional approach, we investigate the bifurcation phenomena by increasing the parameter  $B$  from the fixed point side.

Noises are found to tilt the competition in favor of the fixed point over the limit cycle. The intrinsic periodicity is suppressed in such a way that the spirals of  $\bar{x}(t)$  and  $\bar{y}(t)$  are damped-out faster if noise amplitude  $D_i$  is larger. Fig. 1 presents a typical situation in which we can see that spiraling amplitudes is compressed and that the relaxation time is reduced. Noises play a dissipative role to suppress the intrinsic mechanism of self-sustaining oscillation.

Detailed studies of the transient properties show that the response of the nonlinear system to the noise perturbation does show somewhat surprising results, especially during the intermediate steps of the transient process preceding to the asymptotic steady state. For some cases, damping of the spiraling trajectory is not so simple as is shown in Fig. 1, the intrinsic periodicity could be enhanced momentarily even though the overall transient is marked by the noise-assisted damping. The noise-enhanced periodicity as shown in Fig. 2 is intrinsic in origin. It occurs only for a short moment of time, and for restricted ranges of noise and rate parameters. This phenomenon is different from the usual stochastic resonance [4,5] in which external periodicity is enhanced by noises. The prediction of stochastic resonance without periodic driving [15] could probably be related to this noise-enhanced periodicity.

Noises cause destabilization effects on limit cycle oscillations [6-9]. when noise strength becomes large enough, the noise-assisted dissipation could destroy the limit cycle trajectory and drive the system to a fixed point attractor. This is related to the noise-induced suppression of periodicity discussed above. In Fig. 3, a perpetual oscillation is damped out by noise in such a way that trajectory relaxes quickly toward a fixed point with  $(\bar{x}^s, \bar{y}^s) \approx (x^s, y^s) = (1, 2)$ . We also observe in both Fig. 2 and Fig. 3 that the spiraling amplitude is compressed considerably while the period becomes roughly half of the deterministic value.

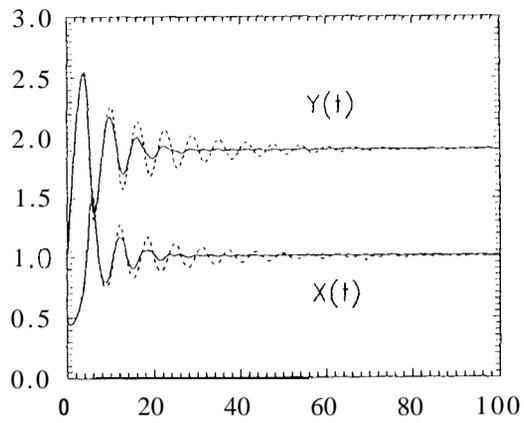


FIG. 1. Noise-assisted damping of  $F(t)$  and  $y(t)$ . Solid and dashed curves stand respectively for noisy and deterministic transient processes. Result are derived for multiplicative noises in Stratonovich interpretation.  $A = 1$ ,  $B = 1.9$  and  $D_B = 0.5$  are assumed.

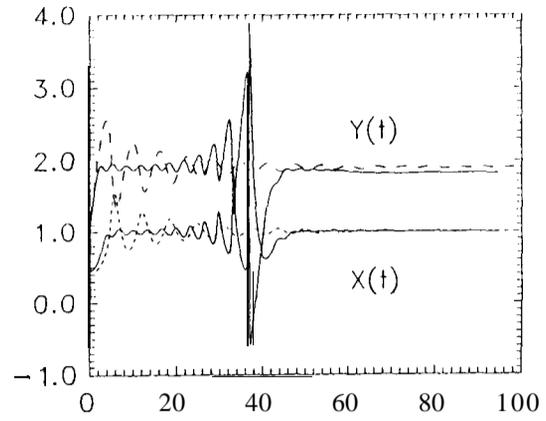


FIG. 2. Noise-assisted damping with intermediate amplification of periodicity. Solid and dashed curves stand respectively for noisy and deterministic transient processes. Result are derived for additive noises with  $A = 1$ ,  $B = 1.9$ ,  $D_1 = 0.5$  and  $D_2 = 0$ .

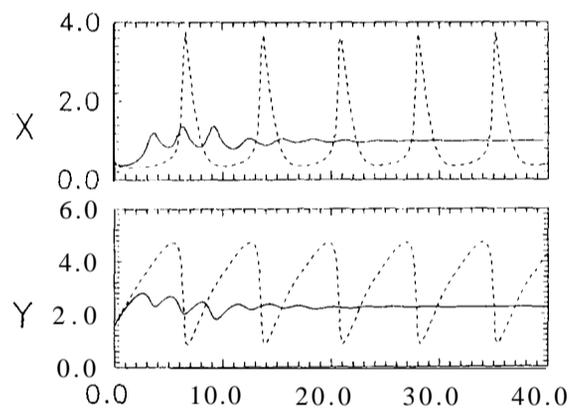


FIG. 3. Noise-induced transition of a limit cycle oscillation to a fixed point. Solid and dashed curves stand respectively for noisy and deterministic transient processes. Result are derived for additive noises with  $A = 1$ ,  $B = 3$ ,  $D_1 = 0.7$  and  $D_2 = 0$ .

## V. Phase diagram of stochastic bifurcation

Bifurcation between fixed point and limit cycle attractors is a result of competition between the intrinsic mechanism of dissipation and self-sustaining oscillation. For a fixed value of  $A = 1$ , the bifurcation parameter  $B$  control the competition of the two distinct mechanism.

When the random elements are taking into consideration, both the competing mechanism and the asymptotic attractors are perturbed. Limit cycle is more vulnerable to noise effects [6-9] than the fixed point is. For the Brusselator model is concerned, we find that noises suppress oscillation and that limit cycle could be damped out at large noise. Therefore, it is reasonable to believe that noises tilt the balance of competition in favor of fixed point. The domain of fixed point could be expanded.

Extensive studies show that transition from a limit cycle to a fixed point occurs only for specific noises with considerably large noise intensities. All the three types of additive noises discussed in Sec. 3 have qualitatively same effects on the transition. The minimum noise amplitudes required to do so are found to be  $D_1^{min} \approx 0.5$ ,  $D_2^{min} \approx 0.4$  and  $D_1^{min} = D_2^{min} \approx 0.3$  respectively for the three noises. These values are estimated at the deterministic bifurcation point at  $B_c = 2$ .  $D^{min}$  becomes larger for larger value of  $B$ , as is shown in Fig. 4. The multiplicative noises are found to weaken and destroy the limit cycle oscillation without transition to a stable fixed point, as was found in other models studies [8,9]. This conclusion is valid for both Ito ( $\nu = 1$ ) and Stratonovich ( $\nu = 2$ ) results.

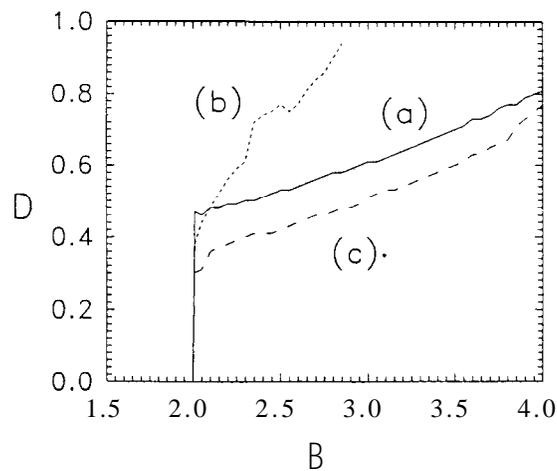


FIG. 4. Phase boundaries of fixed point and limit cycle attractors are plotted for the three types of additive noises with (a)  $D_1 \neq 0, D_2 = 0$ . (b)  $D_1 = 0, D_2 \neq 0$  and (c)  $D_1 = D_2 \neq 0$ .  $A = 1$  is assumed. The deterministic counterpart is the Hopf bifurcation point, which is located on the horizontal axis with  $B = B_c = 2$  and  $D = 0$ .

Transition from fixed point to limit cycle is usually observed by increasing  $B$  to pass through the bifurcation point at  $B_c = 2$ . With appropriate noises, a fixed point attractor could remain to be a stable one even if  $B > 2$ ; transition to limit cycle requires a larger value of  $B$ . This results in a postponement of the Hopf bifurcation, and an expansion of phase domain of fixed point in the  $D - B$  space as is shown in Fig. 4. The phase boundary separating the two competing attractors is also the *bifurcation* locus for the stochastic Hopf bifurcation. The deterministic counterpart is the usual *bifurcation point* locating on the horizontal axis with  $B = 2$  and  $D = 0$ .

## VI. Conclusions and discussions

By employing approximation schemes, we successfully derive the transient processes leading to the asymptotic attractors, and obtain the stochastic bifurcation characteristics. Since fluctuations remain small all the time, our results and conclusions are believed to be reliable even when the system is close to the bifurcation point.

Our results confirm the finding that Hopf bifurcation is always postponed by noises [10,11]. Instead of analyzing the steady state properties of  $P(x, y; t)$  by simulating the dynamics through digital or analog means, we monitor the transient characteristics by deriving the moments of  $P(x, y; t)$ . In doing so we are able to conclude that the origin of stochastic bifurcation is the noise-assisted damping which suppresses intrinsic periodicity.

Noise effect is an interesting topics since noises play constructive as well as destructive roles. Nonlinearity of the systems, which provides rich scenario in deterministic dynamics, causes interesting response to noisy perturbations. The observation of the periodicity enhancement during the intermediate stage of the damping process is indeed striking since this might be related to the stochastic resonance without external driving [15]. The transient processes, which constitute an essential part of nonequilibrium phenomena, deserve more efforts especially in noisy systems.

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